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Professor S.P. Singh
on his 70th Birthday

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The mathematical content of this journal is indexed in the Society's biweekly **Current Mathematical Publications** and is reviewed, among others, by **Mathematical Reviews** (U.S.A), **Zentralblatt für Mathematik** (Germany) and **Applied Mechanics Reviews** (U.S.A.).]

All communications regarding subsriptions, order for back volumes, membership of the *Vijñāna Parishad of India*, change of address, etc. and all books for review, should be addressed to:

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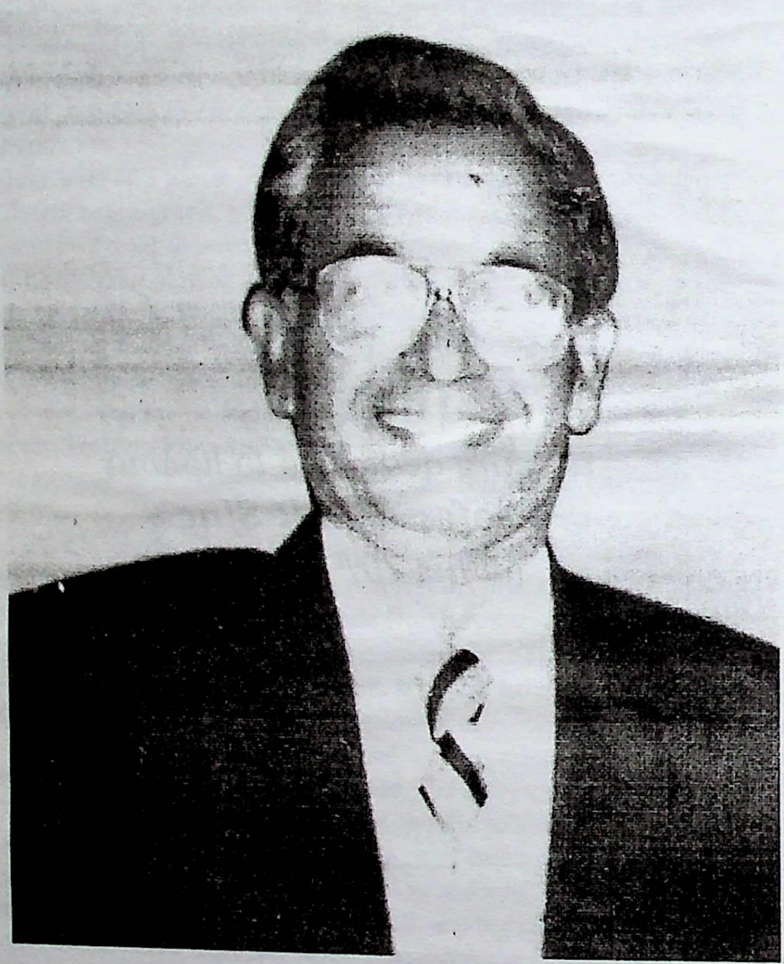
This Volume of

Jñānābha

is being dedicated to honour

Professor S.P. Singh

on his 70^{th} Birthday



Professor S.P. Singh
(Born : January 27, 1937)

Jñānābha, Vol. 37, 2007
(Dedicated to Honour Professor S.P. Singh on his 70th Birthday)

PROFESSOR S.P. SINGH : AS A MAN AND MATHEMATICIAN

By
R.C. Singh Chandel
Secretary, Vijñāna Parishad of India,
D.V. Postgraduate College, Orai-285001, U.P., India

Professor S.P. Singh needs no introduction. He is a remarkable man, a towering mathematician, a major topologist and well known eminent figure of non-linear Analysis: Approximation Theory and Fixed Point Theory. We feel great honoured to introduce Professor S.P. Singh, who is very well actively associated with Jñānābha family since 1980. He was honoured by "***Distinguished Service Award***" ***of*** "**Vijñāna Parishad of India**" in **Silver Jubilee Conference** of the Parishad held at **Parishad Head Quarters :** *D.V. Postgraduate College, Orai, U.P., India* in May 1996, for his distinguished services rendered to Mathematics and Vijñāna Parisahd of India.

It is modest tribute at the occasion of his 70th Birthday Celeberations, when he has been elected as ***Honorary Fellow*** of Vijñāna Parishad of India (F.V.P.I.) during **12th Annual Conference of VPI** held at J.N.V. University, Jodhpur (October 25-27, 2007).

Professor Singh, the ocean of generosity has inspired and helped many students, teachers and collegues to accept multidimensional challenges of physical realty.

PROFESSOR S.P. SINGH WITH HIS FAMILY MEMBERS

Ancestral (Native) Place (Exact Place of Birth)	: Gopiganj, Varanasi, Uttar Pradesh, India
Date of Birth	: Jan 27, 1937
Father	: Late Sri Mahadeo Singh
Mother	: Late Smt. Dharma Devi
Brother	: Brahm Din Singh
	: Uma Shankar Singh
	: Late Professor K.L. Singh (Univ. of Fayetteville State University, N.C. U.S.A.)
Wife	: Smt. Suman Singh
Son	: Manoj Singh
Daughter-in-law	: Tanuja (D/o Dr. Panjab Singh Former V.C., B.H.U.)
Daughter	: Indu Sudha (Anesthsia)
Son-in-law	: Abhay Kumar (Psychiatric)
Grand Son	: Shivank (S/o Manoj)
Grand Daughter	: Priyanka (D/o Indu Sudha)

PROFESSOR S. P. SINGH AT A GLANCE

Full Name	: **Sankatha Prasad Singh**
Address	: 750 Hedgerow Place, London, ON, Canada N6G5H9
	: Physics Department University of Western Ontario, London, Canada, N6G3K7
Telephone	: (519) 6612111 Ext. 86455 (Office)
	(519) 6727540 (Home)
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Email	: spsingh@math.mun.ca
	ssingh@morgan.ucs.mun.ca
	ssingh2@uwo.ca ssingh55@rogers.com

Qualifications

B.Sc. : 1957(Agra University), Agra, India)

M.Sc. : 1959 (Banaras Hindu University), Varanasi, India)

Ph.D. : 1964 (Banaras Hindu University, Varanasi, India)

Teaching Experience

Banaras Hindu University	Lecturer	1959-63
Universi.v of Illinois, USA	Lecturer	1963-64
Wayne state University, USA	Asstt. Professor	1964-65
University of Windsor, Canada	Asstt. Professor	1965-67
Memorial University, Canada	Assoc. Professor	1967-72
Memorial University, Candad	Professor	1972-2002
Shawnee State University, USA	Adjunct Professor	(shorter visit-a few times: 2000-02)
University of Western Ontario,	Visiting Professor	2002-continued

Membership

American Math. Soc, Canadian Math. Soc, Allahabad Math. Soc., Indian Math. Soc., Inst. of Math. & Its Applications, Vij ñāna Parishad of India, Bharat Ganita Parishad, Poorvanchal Academy of Sciences, Unione Mat Italiana, Indian Academy of Mathmatics, Soc. of Special Functions & Applications.

Fellowship

Fellow of the Inst. of Math. & Its Applications, (*FIMA (UK)*)

Fellowship of the National Academy of Sciences, India (*F.N.A. Sc*)

Fellowship of the Allahabad Math. Society (*FALMS*)

Fellowship of Vij ñāna Parishad India (*FVPI*)

Editor

Bulletin, Vij ñāna Parishad of India

Foreign Editor

Journal of Natural and Physical Sciences

Member

Council of the Allahabad Mathematical Society

Member Editorial Board

Ganita (Bharat Ganita Parishad), Poorvanchal Academy of Science, Jñānābha (Journal Vijñāna Parishad of India), Epsilon Journal of Mathematics, Far East J. of Math Sci., Intern. J. of Sci. & Eng., Journal of Inequalities in Pure and Applied Math (*JIPAM*), Lecture Notes: Allahabad Math Society, Vikram J. of Mathematics, Varahmihir J. of Math Sci., Nigerian J. of Math Sci., Southeastern Asian J of Math & Math Sciences, Australian J. of Math Analysis & Appl. (*AJMAA*), Oriental J. of Mathematics, Global J. of Math & Math Sc. (*GJMMS*), Intern. J. of Pure & Appl. Math Sciences (*IJPAMS*), Intern. J. of Math Sciences., Chhatisgarh J. of Mathematics and engineering, . Journal of Fixed point Theory

Member Advisory Board

i) Indian Academy of Mathematics
ii) Nigerian Journal of Scientific Research
iii) International J. of Special Functions and Applications

Member Research Board of Advisors

The American Biographical Institute

Reviewer

Mathmatical Reviews, USA;
Zentrablatt für Mathematik, Germany

Director-Nato-Advanced Study Institute

i) Approximation Theory - 1982-83
ii) Nonlinear Functional Analysis - 1984-85
iii) Approximation Theory, Spline Functions & Applications- 1990-91
iv) Approximation Theory, Wavelets & Applications - 1994-95

Project Leader

NATO- International Research Group - 1980-82.

Research Field

Approximation Theory, Fixed point Theory (Nonlinear Analysis), Variational Inequalities, Special Functions, Mathematical Economics.

Research, in part, has been supported by *NSERC* (Natural Sciences and Engineering Research Council, Canada) *NATO* (North Atlantic Treaty Oraganization), *CNR* (Italian Research Council), Canadian Math Congress, University Grants Commission, India and Memorial University, Secretary of State, Government of Canada.

Books and Proceedings

1. *Lecture Notes on Fixed-point Theorems in Metric and Branch Spaces*, Matscience, Madras (1974), p. 112.

2. *Proc. Intern. Conf. on Nonlinear Analysis & Applications,* Marcel Dekker, New York (1982). p. 465 (with J. Burry).
3. *Proc. Conf. on Topological Methods in Nonlinear Analysis,* Contemp. Math., American Mathl. Soc. (1983), p. 218 (with S. Thomeier and B. Watson).
4. *Proc. NATO-ASI on Approximation Theory & Spline Functions,* D. Reidel Publ. Co. (1984), p. 485(with J. Burry and B. Watson).
5. *Proc. Intern. Conf. on Approximation Theory*, Pitman Publ. Co., London, UK (1985), P. 264.
6. *Proc. NATO-ASI on Nonlinear Functional Analysis*, D. Reidel Publ. Co. (1986), p. 418.
7. *Proc. Intern. Conf. on Solutions of Operator Equations & Fixed Points*, The Math. Soc. Res. Inst. Korea (1986), p. 250 (with V.M. Seghal & J. Burry).
8. *Banach Contraction Principle and its Applications to Integral Equations,* Memorial University of New Foundland, C-CORE (1988), p. 42 (with J. Walsh).
9. *Proc. NATO-ASI on Approximation Theory, Spline Functions and Applications,* Kluwer Academic Publishers (1992), P. 475 (with A. Carbone, R. Charron, and B. Watson).
10. *Proc. NATO-ASI on Approx. Theory, Wavelets and Applications*, Kluwer Academic Publishers (1995), p. 572(with A. Carbone and B. Watson).
11. *Fixed point Theory and Best Approximation:* The KKM-Map Principle, Kluwer Academic Publishers (1997), p. 220 (with B. Watson and P. Srivastava).
12. *Proc. International Conf. on Nonlinear Analysis and Its Applications,* Nonlinear Analysis Forum 6 (2001) p. 275 (with B. Watson).

External Lectures

2006: Banaras Hindu University
4 lectures (Science College, Math. Engineering College, Faculty of Management, and Women's College) Math.
University of Allahabad
UP Open University Allahabad.
Organized a Special Session in the American Math Society and presented a paper. October, 2006, Cincinnati, USA

2005 : HRI, India
IIT Roorkee, India,
Intern. Conf. Cetraro, Italy

2004 : University of Calabira, Italy
University of Perugia,
Iniversity of Roma 3,
ICTP, Trieste Italy

2003 : University of Perugia, Italy

University of Calabria, Italy

2002 : Aligarh Muslim University India
Indian Math. Society Meeting, Jan. 27-31.
Carleton University, April 5
University of Calabria, Math Dept. June 18
Univ. of Calabria, Centre of Num. Analysis, June 19, Italy

2001 University of Calabria, Italy International Conference on Functional Analysis Methods in Economics and Finance, Italy, June 28-July 2, 2001.

2000 University of Calabria, Italy
University of Rome, Italy
University of Torino, Italy
University of Florence, Italy
University of Siena, Italy
University of Perugia, Italy

1999 Shawnee State University, Ohio, USA
University of Calabria
University of Torino

1998 Shawnee State University (4 lectures)
University of Torino Italy (2 lectures)
University of Bari, Italy
University of Calabria, Italy (2 lectures)

1997 University of Rome, Italy
University of Bari, Italy
University of Calabria, Italy
University of Torino, Italy
American Math Society Meeting, Detroit
R.D. University, Jabalpur, India
Allahabad Math Society

1996 University of Seina, Italy
University of Perugia, Italy
University of Salerrno, Italy
University of Rome, Italy
University of Torino, Italy (Series of 4 lectures)
Silver Jubilee Conference, Vijñāna Parishad of India
Delhi University, Delhi, India

1995 FAMA=95, Cosenza, Italy
University of Calabria, Italy
University of Perugia, Italy
University of L=Aquila, Italy

Italian Research Council (CNR), Naples
University of Catania, Italy
Banaras HIndu University, India
Mehta Research Institute, Allahabad, India
Delhi University, Delhi, India
IIT Delhi, India
Panjab University, Chandigarh
Gurukul Kangari University, Hardwar
The National Seminar in Math, Bhopal
Approximation Theory Conference, Taxas

1994 NATO-ASI, Maratea, Italy
University of Calabria, Italy
University of Torino, Itlay (2 lectures)
University of Perugia, Italy
University of Rome, Italy
Royale Military College, Kingston, Ontario (Colloquium)

1993 Conf. del Seminario Mat., University of Torino, Italy
University of Torino, Italy
Universita di Calabria, Italy
University of Firenze, Italy
University of Rome II, Rome, Italy
University of Rome, Italy
Universita di Calabria, Italy
Dalian Institute of Technology, China (Series of 4 lectures)
Beijing University, China

1992 Lunknow University, India
Allahabad Math Society, India
Udai Pratap College, India
Banaras Hindu University, India
University of Torino, Italy (Series of 4 lectures)
University of Calabria, Italy
University of Napoli, Italy
University of Rome, Italy

1991 University of Bari, Italy
Italian Research Council (CNR), Napoli, Italy
University of Milano, Italy
University of Florence, Italy

1990 Banaras Hindu University, India
University of Messina, Italy

University of Calabria, Italy
Scuola Intern. Superiore di Studi Avanzati (SISSA), Trieste, Italy
International Centre of Theoretical Physics (ICTP), Trieste, Italy

1989 Allahabad Math Society, India
Banaras Hindu University, India
University of Calabria, Italy

1988 Fayetteville State University, North Carolina, USA
Carleton University, Ottawa, Ontario
Laval University, Quebec City, Quebec
Lakehead University, Thunder Bay, Ontario
Concordia University, Montreal, Quebec
Concordia University, Montreal, Quebec
University of Milano, Italy
University of Calabria, Italy
University of Napoli, Italy
University of Catania, Italy
University of Delhi, India
IIT Bombay, India (series of 4 lectures)
Mehta Research Institute, Allahabad, India (series of 3 lectures)

1987 University of Prince Edward Island, Charlottetown, Prince Edward Island
University of Milano, Italy
University of Calabria, Italy
University of Napoli, Italy
IIT Bombay, India
Mehta Tesearch Institute, Allahabad, India

Conferences/Meetings Attended

2005 Nonlinear functional analysis applications in economics and Finance, Cetraro , Italy

2003 ISAAC Conference, Toronto

2002

1. Indian Mathematical Soc. Meeting, Aligarh, India January 27-31,
2. Vij ñāna Parishad of India (VPI) Meeting, Delhi, India Feb. 24-26
3. D.S.T. meeting, Allahabad University, Feb. 12-14

2001 International Conference on Functional Analysis Mathods in Economics and Finance

2000

1. Organized a Special Session in Toronto AMS Meeting, September 2000.
2. Presented a paper in Toronto Meeting.
3. Attended Allahad Math. Soc. Council Meeting.

4. Attended AMS Annual Meeting, Washington, DC, January 2000.

1999 Organized a conference on Nonlinear Analysis and Applications, Canadian Mathematical Society Summer Meeting, May.

1998 American Mathematical Society, Louisville, Kentucky March.

1997 American Mathematical Society, Detroit, April.

1995

1. 8th Texas Conference on Approximation Theory, January 7-12.
2. National Seminar in Mathematics, Bhopal, India, March 13-14.
3. International Conference on Functional Analysis, Methods and Applications, Cosenza, Italy, May 27-June 2.

1994 NATO-ASI on Approximation Theory, Wavelets and Applications, Italy, May 16-26.

1993

1. Conference on Functional Analysis and Applications, Milano, Italy, May, 10-14.
2. Topological Analysis Workshop on Degree Theory, Singularity and Variations, Rome, May 28-31.
3. NATO-ASI, Montreal, July 26-August 6.
4. NATO-ASI, Funchal, Madeira, Portugal, August 6-21.

1992 Allahabad Math Society, February 7-8.

1991

1. NATO-ASI, Maratea, Italy, April 28-May 09.
2. Canadian Mathematical Society Summer Meeting, Sherbrooke, May 29-June 02.
3. Research Workshop on Topological and Variational Methods, Sherbrooke, May 28-June 07.
4. Second International Conference on Fixed point Theeory, Halifax, June 9-14.

1990

1. NATO-ASI on Shape Optimization and Free Boundaries Montreal, June 27-July 13.
2. Conference on Recent Developments in Analysis, Manifolds and Application, Banaras Mathematical Society, Varanasi, India, January 3-5.
3. Ontario Mathematical Meeting, St. Catherines, April 18-20

1989

1. Indian Mathematical Society Annual Meeting, Delhi, December 26-30.
2. International Conference on Approximation Theory, Jabalpur, India, December 15-18.
3. APICS Meeting, Sackville, New Brunswick, October 27-28.
4. International Conference on Functional Analysis, Cetraro, Italy, June 26-30.

5. International Conference on Fixed Point Theory, Marseille, France, June 5-9.
6. NATO-ASI, Columbus, Ohio, May 21-June 3.
7. American Mathematical Society Meeting, Chicago, Illinois, May 19-20
8. Allahabad Mathematical Society, Allahabad, January 20.

1988 1. American Mathematical Society Meeting, Atlanta, January 12.
2. NATO-ASI, Istanbul, Turkey, August 15-18.

1987 1. Canadian Mathematical Society Meeting, Kingston, Ontario, June.
2. Learned Societies Meetings, Hamilton, Ontario, June.
3. American Mathematical Society Meeting, Salt Lake City, Utah, (chaired one session), August.
4. Allahabad Mathematical Society, November.

Publications

1. S.P. Singh, E. Tarafdar and B. Watson, A generalized Urysohn imbedding and Tychonoff fixed point theorem in topological space, *Indian J. Pure & Appl. Math.*, **34** (2003) 227-673.
2. E.Tarafdar, S.P. Singh and B. Watson, Fixed point theorems for some extensions of contraction mappings in uniform spaces, *J. Math Sci.* **1** (2002) 53-61.
3. J. Li and S.P. Singh, An extension of Ky Fan=s best approximation theorem. *Nonl. Anal. Forum,* **6** (2001), 163-170.
4. M. S. R. Chowdhury, E. Tarafdar, S.P. Singh and B. Watson, Generalized quasi-variational inequalities for Hemi- continuous operators on non-compact sets, *Nonl. Anal. Forum* **6** (2001), 79-90.
5. P. Kumar, S.P. Singh, and S. S. Dragomir, Some inequalitties involving beta and gamma functions, *Nonl. Anal. Forum* **6** (2001), 143-150.
6. G. Isac, V. M. Sehgal and S. P. Singh, An alternate version of a variational inequality, *Indian J. Math.,* **41** (1999), 25-31.
7. S. P. Singh, E. Tarafdar and B. Watson, A generalized fixed point theorem and equilibrium point of an abstract economy, *J. Comp. Appl. Math.,* **113** (2000), 65-71.
8. T. D. Narang and S. P. Singh, Best coapproximation in metric linear spaces, *Tamkang J. Math.* **30** (1999), 241-252.
9. J. Li and S. P. Singh, An extension of Fan=s theorem, *G.K. Vignana Patrika* **1** (1998), 133-146.
10. S. P. Singh, Best approximation and fixed point theorems, *Proc. Nat. Acad. Sci. India*, **LXVII** (1997), 1-27 (survey paper).
11. S. P. Singh, E. Tarafdar and B. Watson, Variational inequalities and applications, *Indian J. Pure Applied Math.* **28** (1997), 1083-1089.

12. T. D. Narang and S. P. Singh, Best coapproximation in locally convex spaces, *Tamkang J. Math.* **28** (1997), 1-5.
13. S. P. Singh, E. Tarafdar and B. Watson, Variational inequalities for a pair of Pseudomonotone functions, *Far East J. Math Sci. Special Volume* (1996), 31-52.
14. V. M. Sehgal and S. P. Singh, Coincidence thorems for paracompact sets, *J. Indian Math Soc.*, **64** (1997), 97-101.
15. D. V. Pai and S. P. Singh, On *p*-centers and fixed points, *Proc. Intern. Conf. on Approximation Theory and Its Applications*, New AGE Intern. Publishers, Ed. G. S. Rao (1996), 87-98.
16. A. Carbone and S. P. Singh. On Fan=s best approximation and applications, *Rend. Sem. Mat. Univ. Pol. Torino,* **54** (1996), 35-52.
17. V. M. Sehgal and S. P. Singh. The best approximation and an extension of a fixed point theorem of F. E. Browder, *Intern. J. Math and Math. Sco.*, **18** (1995), 745-748.
18. S. P. Singh, On Ky Fan=s best approximation and fixed point theorems, *J. Nat. & Physical Sci.*, **9-10** (1995-96), 25-42.
19. S. P. Singh, On Fan=s best approximation theorem, *Proc. Intern. Conf. on Approximation Theory VII*, World Scientific Publ. Co. Eds. Chui and Schumaker (1995), 529-536.
20. S. Park, S. P. Singh and B. Watson, Some fixed point theorems for composites of acyclic. maps., *Proc. Amer. Math Soc.*, **25** (1994), 1151-1158.
21. S. Park, S. P. Singh and B. Watson, Remarks on best approximations and fixed points, *Indian J. Pure Appl. Math Soc.*, **25** (1994), 459-462.
22. S. P. Singh and B. Watson, Best approximation and fixed point theorems, *Proc. NATO-ASI,* Kluwer Academic Publishers (1994), 285-294.
23. G. Marino, P. Pietramala and S. P. Singh, Convergence of approximating fixed points sets. for multivalued mappings, *J. Math. Sci.,* **28** (1994), 117-130.
24. A. Carbone and S. P. Singh, On some fixed point theorems in Hilbert spaces, *Dem. Math.* **XXVI** (1993), 649-656.
25. S. P. Singh and B. Watson, On convergence results in fixed point theory, *Rend. Sem. Math. Univ. Pol. Torino,* **51** (1993), 73-91.
26. S. P. Singh and S. Sessa, Application of the KKM-principle to Prolla type theorems, *Tamkang J. Math.* **23** (1992), 279-287.
27. V. M. Sehgal and S.P. Singh, On best simultaneous approximations in normed linear spaces. *Proc. NATO-ASI*, Kluwer Academic Publishers (1992), 463-470.
28. V. M. Sehgal and S.P. Singh and B. Watson, Best approximation and fixed points, *Bull. Allahabad Math. Soc.,* **6** (1991), 11-27.
29. V. M. Sehgal and S.P. Singh and G. C. Gastl, On a fixed point theorem for

multivalued Maps, *Proc. Intern. Conf. On Fixed Point Theory and Applications*, Eds. Baillon and Thera, Pittman Publishers (1991), 377-382.

30. S. P. Singh and P. V. Subrahmanyam, A note on fixed points and coincidences. *J. Poorvanchal Acad. Sci.*, **2** (1991), 1-5.
31. V. M. Sehgal, and S. P. Singh, A Ky Fan type theorem and its applications, *Progr. Math.*, **24** (1990), 13-18.
32. V. M. Sehgal, S. P. Singh and J. H. M. Whitfield, KKM-maps and fixed point theorems. *Indian J. Math.*, **32** (1990), 289-296.
33. S. Sessa and S. P. Singh, Applications of KKM-Map principle, *J. Poorvanchal Acad. Sci.*, **1** (1990), 1-6.
34. V. M. Sehgal and S.P. Singh, A theorem on best approximation, *Numer. Funct. Anal. Optimiz.* **10** (1989), 181-184.
35. A. Carbone, B. E. Rhoades and S.P. Singh, A Fixed point theorem for generalized contraction map, *Indian J. Pure Appl. Math.*, **20** (1989), 543-548.
36. D. Roux and S. P. Singh, On some fixed point theorems, *Intern J. Math & Math Sci.* **12** (1989), 61-64.
37. V. M. Sehgal and S. P. Singh, Fixed point theorem for contraction type mappings in convex spaces, *Jñānābha*, **19** (1989), 11-14.
38. D. Roux and S. P. Singh, On a best approximation theorem, *Jñānābha*, **19** (1989), 1-9.
39. V. M. Sehgal and S.P. Singh, A generalization to multifunctions of Fan=s best approximation theorem, *Proc. Amer. Math. Soc.*, **102** (1988), 534-537.
40. S.P. Singh, On Ky Fan=s theorem and its applications I, *Rend. Sem. Math. & Fis. di*
41. A. Carbone and S. P. Singh. Fixed point theorems for Altman type mappings. Indian J. *Pure Appl. Math.* **18** (1987), 1082-1087.
42. V. M. Sehgal and S.P. Singh and R. E. Smithson, Nearest ponts and some fixed point theorems for weakly compact sets, *J. Math. Anal. Appl.*, **128** (1987), 108-111.
43. S. Massa, D. Roux and S. P. Singh, Fixed point theorems for multifuctions, *Indian J. Pure Appl. Math.* **18** (1987), 512-514.
44. S. P. Singh, On Ky Fan =s theorem and its applications II,. *Nonlinear Analysis* (Ed. T.M. Rassias). World Sci. Publ. Co. (1987), 527-37.
45. V. M. Sehgal and S.P. Singh, On rendom approximation and a random fixed point theorem, *MSRI Korea Publ.*, **1** (1986), 99-102.
46. S. P. Singh and B. Watson, On approximating fixed points, *Proc. Symp. Pure Math., Amer. Math. Soc.*, **45** (1986), 393-395.
47. S. P. Singh, S. Massa and D. Roux, Approximation techniques in fixed point theory, *Rend. Sem. Mat. Fis. di Milano*, **LIII** (1983), 165-172.

48. V. M. Sehgal and S.P. Singh, On rendom approximation and a random fixed point theorem for set-valued mappings, *Proc. Amer. Math. Soc.*, **95** (1985), 91-94.
49. V. M. Sehgal and S.P. Singh, A theorem on the minimization of condensing multifunction and fixed points. *J. Math. Anal. Appl.*, **107**(1985), 96-102.
50. H. M. Srivastava and S. P. Singh, A note on the Widder transform related to the Poisson integral for a half plane. *Inter. J. Math. Edu. Sci. Tech.* **16** (1985), 675-377.
51. V. M. Sehgal and S.P. Singh, A Variant of a theorem of Ky Fan, *Indian J. Math. Soc.*, **25** (1984), 171-174..
52. V. M. Sehgal and S.P. Singh, A fixed point theorem in reflexive Banach spaces, *Math. Sem. Notes,* **11** (1983), 81-82.
53. S. P. Singh and B. Watson, Proximity maps and fixed points, *J. Approx. Theory,* **39**(1983), 72-76.
54. V. M. Sehgal, S.P. Singh and B. Watson, A coincidence theorem for topological vector spces, *Indian J. Pure Appl. Math.,* **14** (1983), 565-566.
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2002 MUNFA pension com., MUNFA scolarship com., Board of Regents pension Com.

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1994. Senate Committee on Copyright, MUNFA, Scholarships, Pension, Allahabad Mathematical Society Council.

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1991 Promotion & Tenure Committee, Search Committee

1990 Promotion & Tenure Committee, Search Committee

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TEACHING RECORD

2001	S	Math 1090, 2050
	W	Math 1000, 3260
	F	Sabbatical
2000	W	Math 1090
	F	Sabbatical
1999	W	Math 2000, 3260
	F	Math 1001, 1081
1998	S	Math 2000, 3202
	F	Math 1081, 3260
1997	F	Math 3210, 3301
	S	Math 2000, 3202
1996	F	Math 2000, 3101/3301
	S	Math 2000, 3260
1995	F	Math 3210, 3301
	W	Sabbatical
1994	W	Math 2000, 3260
	S	Math 1080,1001
	F	Sabbatical
1993	W	Math 1081, 4310
1992	F	Math 1080, 3210
	S	Math 2000, 3260
1991	F	Math 1081, 3201

	W	Math 1081, 3210
1990	F	Math 1081,2001
	W	Math 1080,2000
1989	F	Math 1001, plus labs

THESES SUPERVISED

MASTER'S THESES

C.W. Norris	Fixed point theorems in generalized metric spaces and applications
W. Russell	Fixed point theorems in uniform spaces
L. S. Dube	Nonexpansive mappings and applications
P. S. Cheema	Fixed point theorems in metric spaces and applications
F. Hynes	Iterated contraction mappings and applications
W. Ivimey	Multivalued contraction mappings and appilcations
B. Holden	Sum of nonlinear operators and fixed points
M. I. Riggio	Measure of noncompactness, densifying mappings and fixed points
S. Deb	Quasibounded mappings, fixed point theorems and applications
R. K. Yadav	Geometry of Banach spaces and some fixed point theorems
M. Veitch	Fixed point theorems for mappings with a convexity condition
G. Collins	Extension of some fixed point theorems in metric spaces
A. Robertson	Convergence of the sequence of successive approximations
B. Meade	Extension of some fixed point theorems and applications

HONOR'S THESES

R. Blair	Banach contraction principle
D. Payne	Conformal mappings and applications
D. Ryan	Contraction mapping principle and applications

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A CORRECTION TO AN EXAMPLE OF RENU CHUG AND SANJAY KUMAR FOR WEAKLY COMPATIBLE SELF-MAPS

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ABSTRACT

A correction is suggested in an example of Renu Chug and Sanjay Kumar, in attempting to disprove the converse of the statement that every compatible pair of self-maps is weakly compatible.

2000 Mathematics Subject Classification : 54H25
Keywords and Phrases : Weakly compatible self-maps.

1. Introduction. Let (X,d) denotes a metric space. As a generalization of commuting self-maps, Gerald Jungck [2] defined self-maps S and A on X to be compatible if $\lim_{n\to\infty} d(SAx_n, ASx_n) = 0$ whenever $\langle x_n \rangle_{n=1}^{\infty}$ is such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. It is easy to see that if self-maps S and A on X are compatible, then $ASx=SAx$ wheneer $x \in X$ is such that $Ax = Sx$. Self-maps which commute at their coincidence points are known as weakly compatible [3]. In an attempt to disprove the converse of the statement that every compatible pair is weakly compatible, Renu Chug and Sanjay Kumar [1] considered self-maps:

$$Ax = \begin{cases} x & (x = 2 \text{ or } x > 5) \\ 6 & (2 < x \le 5) \end{cases} \quad \text{and} \quad Sx = \begin{cases} x & (x = 2) \\ 12 & (2 < x \le 5) \\ x - 3 & (x > 5). \end{cases} \text{for all } x \in X, \qquad (1)$$

where $X=[2,20]$ with usual metric $d(x,y) = |x - y|$ for all $x, y \in X$. It was claimed that the mappings are not compatible.

The following lines reveals that their claim is not true. The maps A and S are, in fact, compatible. For,

$$d(Sx, Ax) = \begin{cases} 0 & (x=2) \\ 6 & (2 < x \le 5) \\ 3 & (x > 5). \end{cases} \text{ and } d(SAx, ASx) = \begin{cases} 0 & (x = 2 \text{ or } x > 8) \\ 9 & (2 < x \le 5) \\ 9 - x & (5 < x \le 8). \end{cases} \quad (2)$$

so that $d(Sx_n, Ax_n) \to 0$ as $n \to \infty$ whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is such that $x_n = 2$ for all n and hence $d(SAx_n, ASx_n) \to 0$ as $n \to \infty$.

However, if we redefine A as $Ax = \begin{cases} 2 & (x = 2 \text{ or } x > 5) \\ 6 & (2 < x \le 5) \end{cases}$ a routine computation reveals that A and S are not compatible. In this case, $x=2$ is the only coincidence point for A and S at which they commute and hence (A,S) is weakly compatible.

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A NOTE ON PAIRWISE SLIGHTLY SEMI-CONTINUOUS FUNCTIONS

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ABSTRACT

In this paper we introduce concept of pairwise slightly semi-cintinuous function in bitopological spaces and discuss some of the basic properties of them. Several eexamples are provided of illustrate behaviour of these new classes of functions.

2000 Mathematics Subject Classification : 54E55.
Keywords and Phrases : (i,j) clopen set, pairwise slightly continuous, pairwise slightly semi-continuous, pairwise almost semi-continuous, pairwise semi θ-continuous, pairwise weakly semi-continuous, pairwise s- closed, pairwise ultra regular.

1. Introduction. Kelly[5] initiated the systematic study of bitopological spaces. A set equiped with two topologies is called bitopological space. Continuity play an important role in topological and bitopological spaces. In 1980, Jain [4] introduced the concept of slightly continuity in topological spaces. Recently Nour [10] defined a slightly semi-continuous functions as a generalization of slightly continuous function using semi-open sets and investigated its properties. In 2000, Noiri and Chae [9] introduced a note on slightly semi-continuous functins in topological spaces.

The object of the present paper is to introduce a new class of functions called pairwise slightly semi-continuous functions. This class contains the class of pairwise continuous functions and that of pairwise semi continuous functions. Relations between this class and other class of pairwise continuous functions are obtained.

Throughout the present paper the spcaes X and Y always represent bitopological spaces (X,P_1,P_2) and (Y,Q_1,Q_2) on which no seperation axioms are assumed. Let $S \subset X$. Then S is said to be (i, j) semi-open [8] if $S \subset P_j-Cl$ (P_i-Int(S)) (where P_j–$Cl(S)$ denotes the closure operator with respect to topology P_j and P_i–Int(S) denotes the interior operator with respect to topology P_i, $(i, j=1,2, i\neq j)$ and its complement is called **(i, j) semi-closed.** The intersection of all (i, j)

semi-closed sets containing S is called the **(i, j) semi-closure** of S and it will be denoted by $(i, j)\ s\ Cl(S)$. A subset S is said to be **(i, j) semi-regular** if S is both (i, j) semi-open and (i, j) semi closed. A subset S is said to be **(i, j) semi θ-open** if S is the union of (i, j) semi-regular sets and the complement of a (i, j) semi θ-open set is called **(i, j) semi θ–closed**. A subset S is said to be **(i, j) clopen** if S is P_i–open and P_j–closed set in X.

In this note we denote the family of all (i, j) semi-open (resp. P_i–open, (i, j) semi-regular and (i, j) clopen of (X,P_1,P_2) by $(i, j)SO(X)$(resp. P_i-open(X),$(i, j)\ SR(X)$ and $(i, j)CO(X)$), and denote the family of (i, j) semi-open (resp. P_i-open, (i, j) semi-regular and (i, j) clopen) set of (X,P_1,P_2) containing x by $(i, j)SO(X,x)$(resp. $P_i\ (X,x)$, $(i, j)SR(X,x)$ and $(i, j)CO(X,x)$. $i, j=1,2$. $i\neq j$.

2. Preliminaries.

Definition 2.1. A function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is said to be pairwise-semi continuous [8] $(p.s.c.)$ (resp. pairwise almost semi-continuous) $(p.a.s.C)$[12], pairwise semi θ-continuous (p,s,θ,C) [12] and pairwise weakly semi-continuous $(p.w.s.C.)$ [12]), if for each $x\in X$ and for each $V\in Q_i(y,f(x))$ there exists $U\in(i,j)SO(X,x)$ such that $f(U)\subset V$(resp. $f(U)\subset Q_i-\text{int}(Q_j-Cl(V))$ $f(i,j)sCl(U)\subset Q_j-Cl(V)$ and $f(U)\subset Q_j-Cl(V)$).

Definition 2.2. A function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is called to be pairwise almost continuous $(p.a.C.)$ [2] (resp. pairwise θ-continuous $(p.\theta.C.)$[1], pairwise weakly continuous $(p.w.C.)$[2] if for each $x\in X$ and for each $V\in Q_i$–$(Y, f(x))$, there is $U\in P_i(X,x)$ such that $f(U)\subset Q_i$–Int$(Q_j$–$Cl(V))$ (resp. $f(P_j$–$Cl(U)\subset Q_j$–$Cl(V)$, $f(U)\subset Q_j$–$Cl(V)$).

Definition 2.3. [11] A function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is called slightly semi-continuous $(p.sl.s.C.)$(resp. pairwise slightly continuous $(p.sl.C)$ if for each $x\in X$ and for each $V\in(i, j)CO(Y, f(x))$, there exists $U\in(i, j)SO(X,x)$ (resp. $U\in P_i(X,x)$) such that $f(U)\subset V$. $i, j=1,2$ and $i\neq j$.

A function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is said to be pairwise slightly semi-continuous (resp. pairwise slightly continuous) if inverse image of each (i, j)-clopen set of Y is (i, j) semi-open (resp. P_i–open) in X $i, j=1,2$, $i\neq j$.

The following diagram is obtained in [11] :

$$\begin{array}{ccccccccc}
p.C & \Rightarrow & p.a.C & \Rightarrow & p.\theta.C. & \Rightarrow & p.w.C & \Rightarrow & p.sl.C \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
p.s.C & \Rightarrow & p.a.s.C & \Rightarrow & p.\theta.s.C & \Rightarrow & p.w.s.C & \Rightarrow & p.sl.s.C
\end{array}$$

diagram 1

Remark 2.4. *It was point out in* [11] *that pairwise slightly continuity implies pairwise slighltly semi-continuity, but not conversely. Its counter example are not*

given in it.

Example 2.5. Let $X=\{a, b, c\}$, $P_1=\{\phi, X, \{a\}, \{b\}, \{a,b\}\}$, $P_2=\{\phi, X, \{a\}, \{a,b\}\}$ and let $Q_1=\{\phi, X, \{a\}\}$, $Q_2=\{\phi, Y, \{b,c\}\}$. Then the mapping $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is pairwise slihtly semi-continuous but not pairwise slightly continuous for $f^{-1}(\{a\})$ is (i, j) semi-open and (i, j) semi-closed, but not P_i –closed in (X,P_1,P_2).

Theorem 2.6. *The pairwise set-connectedness and the pairwise slightly continuity are equivalent for a surjective function.*

Proof. A surjection $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is pairwise set-connected if and only if $f^{-1}(F)$ is (i, j) clopen in X for each (i, j) clopen set F of Y. It is easy to prove that a function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is pairwise slightly continuous if and only if $f^{-1}(F)$ is P_1–open in X for each (i, j) clopen set F of Y. Therefore, the proof is obvious.

Theorem 2.7. For a function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ the following are equivalent:

(a) f is pairwise slightly semi-continuous,

(b) $f^{-1}(V)\in(i, j)SO(X)$ for each $V\in(i, j)CO(Y)$,

(c) $f^{-1}(V)$ is (i, j) semi-open and (i, j) semi-closed for each $V\in(i, j)CO(Y)$.

3. Properties of pairwise slightly semi-continuity.

Theorem 3.1. The following are equivalent for a function $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$:

(a) f is pairwise slightly semi-continuous,

(b) For each $x\in X$ and for each $(V)\in(i, j)CO(Y,f(x))$, there exists $U\in(i, j)SR(X,x)$ such that $f(U)\subset V$,

(c) For each $x\in X$ and for each $(V)\in(i, j)CO(Y, f(x))$, there is $U\in(i, j)SO(X,x)$ such that $f((i, j)sCl(U))\subset V$.

Proof. (a) $\Rightarrow$ (b). Let $x\in X$ and $V\in(i, j)CO(Y, f(x))$. By theorem 2.7, we have $f^{-1}(V)\in(i,j)SR(X,x)$. Put $U=f^{-1}(V))$, then $x\in U$ and $f(U)\subset V$.

(b) $\Rightarrow$ (c). It is obvious and is thus omitted.

(c) $\Rightarrow$ (a). If $U\in(i, j)SO(X)$, then $(i, j)sCl(U)\in(i, j)SO(X)$.

Definition 3.2. *A bitopological space (X,P_1,P_2) is called*

(a) **Pairwise semi-T_2**[6] (resp. pairwise ultra Hausdorff or pairwise UT_2) if for each pair of distinct points x, y of X, there exists a P_1–semi-open (resp. P_1–clopen) set U and a P_2–semi-open (resp. P_2–clopen) set V such that $x\in U$, $y\in V$ and $U\cap V=\phi$.

(b) **Pairwise s-normal**[7] (resp.pairwise ultra normal) if for every P_i-closed set A and P_j-closed set B such that $A\cap B=\phi$, there exist $U\in SO(X,P_j)$ (resp $Co(X,P_j)$ and $V\in SO(X,P_i)$(resp.(X,P_i)such that $A\subset U$, $B\subset V$ and $U\cap V=\phi$,where $i, j=1,2$, $i\neq j$

(c) **Pairwise s-closed** (resp.pairwise mildly compact) if every (i,j) semi regular (resp.(i, j) clopen) cover of (X,P_1,P_2) has a finite subcover. $i, j=1,2$ and $i\neq j$.

Theorem 3.3. If $f:(X,P_1,P_2)\to(Y,Q_1,Q_2)$ is a pairwise slightly semi-continuous injection and Y is pairwise UT_2 ,then X is pairwise semi-T_2.

Proof. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Then since f is injective and Y is pairwise UT_2, $f(x_1) \neq f(x_2)$ and there exist, $V_1, V_2 \in (i,j)CO(Y)$ such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $V_1 \cap V_2 = \phi$. By Theorem 2.7, $x_i \in f^{-1}(V_i) \in (i,j)SO(X)$ for $i = 1,2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus X is pairwise semi-T_2.

Theorem 3.4. If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is a pairwise slightly semi-continuous, P_2-closed injection and Y is pairwise ultra normal, then X is pairwise s-normal.

Proof. Let F_1 and F_2 be disjoint (P_1,P_2)-closed subsets of X. Since f is P_2-closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint (Q_1,Q_2)-closed subsets of Y. Since Y is pairwise ultra normal, $f(F_1)$ and $f(F_2)$ are separeted by disjoint P_i–clopen sets V_1 and P_j-clopen V_2, respectively. Hence $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in (i,j)SO(X)$ for $i = 1,2$ from Theorem 2.7 and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus X is pairwise s-normal.

Theorem 3.5. If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is a pairwise slightly semi-continuous surjection and (X,P_1,P_2) is pairwise s-closed, then Y is pairwise mildly compact.

Proof. Let $\{V_\alpha \mid V_\alpha \in (i,j)CO(Y), \alpha \in \nabla\}$ be a cover of Y. Since f is pairwise slightly semi-continuous, by the Theorem 2.7 $\{f^{-1}(V_\alpha) \mid \alpha \in \nabla\}$ is a (i,j) semi-regular cover of X and so there is a finite subset ∇_0 of ∇ such that

$$X = \bigcup_{\alpha \in \nabla_0} f^{-1}(V_\alpha).$$

Therefore,

$$Y = \bigcup_{\alpha \in \nabla_0} V_\alpha \qquad \text{(since } f \text{ is surjective).}$$

Thus Y is pairwise mildly compact.

Theorem 3.6 If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is pairwise slightly semi-continuous and Y is pairwise UT_2, then the graph $G(f)$ of f is (i,j) semi θ-closed in the bitopological product space $X \times Y$.

Proof. Let $(x,y) \notin G(f)$, then $y \neq f(x)$. Since Y is pairwise UT_2, there exists $V \in (i,j)$ $CO(Y,y)$ and $W \in (i,j)$ $CO(Y, f(x))$ such that $V \cap W = \phi$. Since f is pairwise slightly semi-continuous, by the Theorem 2.7 there exists $U \in (i,j)$ $SR(X,x)$ such that $f(U) \subset W$. Therefore $f(U) \cap V = \phi$ and hence $(U \times V) \cap G(f) = \phi$. Since $U \in (i,j)SR(X,x)$ and $V \in (i,j)$ $CO(Y,y)$,

$(x,y) \in U \times V$ and $U \times V \in (i,j)SR(X \times Y)$. Hence $G(f)$ is (i,j) semi θ- closed.

Theorem 3.7. If $f: (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is pairwise slightly semi-continuous and (Y, Q_1, Q_2) is pairwise UT_2, then $A = \{(x_1,x_2) / f(x_1) = f(x_2)\}$ is $)(i,j)$semi -θ closed in the bitopological product space $X \times X$.

Proof. Let $(x_1,x_2) \notin A$. Then $f(x_1) \neq f(x_2)$. Since Y is pairwise UT_2, there exists $V_1 \in (i,j)$ $CO(Y, f(x_1))$ and $V_2 \in (i,j)$ $CO(Y, f(x_2))$ such that $V_1 \cap V_2 = \phi$. Since f is pairwise

slightly semi-continuous, there exist $U_1, U_2 \in (i,j)SR(X)$ such that $x_i \in U_i$ and $f(U_i) \subset V_i$ for $i=1,2$. Therefore, $(x_1,x_2) \in U_1 \times U_2$, $U_1 \times U_2 \in (i,j)\ SR(X \times X)$, and $(U_1 \times U_2) \cap A = \phi$. So A is (i,j) semi θ-closed in bitopological product space $X \times X$.

Definition 3.8. [3] A bitopological(X, P_1, P_2) is said to be pairwise extermally disconnected if P_2 -closure of each P_1-open set of (X, P_1, P_2) is P_1-open.

Lemma 3.9. Let (X, P_1, P_2) be pairwise exteremally disconnected space, then $U \in (i,j)SR(X)$ if and only if $U \in (i,j)CO(X)$; $i, j=1,2$ and $i \neq j$.

Proof. Let $U \in (i,j)SR(X)$. Since $U \in (i,j)SO(X)$,P_j-$Cl\ (U)=P_j$-$Cl(P_i$-Int (U) and so P_i-$Cl(U) \in P_i(X)$. Since U is (i,j) semi-closed , P_i-Int $(U) = U = P_j$-$Cl\ (U)$ and hence U is (i,j) clopen. The convers is obvious.

Theorem 3.10. If $f : (X, P_1, P_2) \to (Y, Q_1, Q_2)$ is pairwise slightly semi-continuous, and X is pairwise exteremally disconnected then f is pairwise slightly continuous.

Proof. Let $x \in X$ and $V \in (i,j)CO(Y,f(x))$. Since f is pairwise slightly semi-continuous by Theorem 2.7, there exists $U \in (i,j)SR(X,x)$ such that $f(U) \subset V$, since X is pairwise exteremally disconnected by the Lemma 3.9,$U \in (i,j)\ CO\ (X)$ and hence f is pairwise slightly continuous.

Difinition 3.11. A function f : $(X, P_1, P_2) \to (Y, Q_1, Q_2)$ is called pairwise almost strongly θ-semi continuous ($p.a$-$st.\theta.s.C.$) (resp. pairwise strongly θ-semi continuous ($p.st.\theta.s.C.$) if for each $x \in X$ and for each $V \in Q_i(Y,f(x))$, there exists $U \in (i,j)SO(X,x)$ such that $f((i,j)\ sCl(U)) \subset (i,j)sCl(V)$ (resp. $f((i,j)sCl(U)) \subset V$).

Theorem 3.12. If f : $(X, P_1, P_2) \to (Y, Q_1, Q_2)$ is pairwise slightly semi-continuous and (Y, Q_1, Q_2) is pairwise extermally disconnected, then f is pairwise almost strogly θ-semi-continuous.

Proof. Let $x \in X$ and $V \in Q_i(Y,f(x))$, then $(i,j)sCl(V) = Q_i$-$Int(Q_j$-$Cl(V))$ is (i,j) regular open in (Y, Q_1, Q_2). Since Y is pairwise exteremally disconnected, $(i,j)\ sCl(V) \in (i,j)\ CO\ (Y)$. Since f is pairwise slightly semi-continuous, by Theorem 3.1, there exists $U \in (i,j)\ SO\ (X, x)$ such that $f((i,j)sCl(U) \subset (i,j)sCl(V))$. So f is pairwise almost stroghly θ-semi-continuous.

Corollary 3.13. [11] If $f:(X,P_1,P_2) \to (Y, Q_1,Q_2)$ is pairwise slightly semi-continuous and (Y, Q_1, Q_2) is pairwise extermally disconnected, then f is pairwise weakly semi-continuous.

Difinition 3.14. A bitopological space (X, P_1, P_2) is called pairwise ultra regular if for each $U \in P_i(X)$ and for each $x \in U$, there exists $O \in (i,j)\ CO\ (X)$ such that $x \in O \subset U$.

Theorem 3.15. If $f:(X,P_1,P_2) \to (Y,Q_1,Q_2)$ is pairwise slightly semi-continuous and (Y,Q_1,Q_2) is pairwise ultra regular, then f is pairwise stroghly θ-semi-continuous.

Proof. Let $x \in X$ and $V \in Q_i(Y,f(X))$. Since (Y, Q_1, Q_2) is pairwise ultra regular, there is $W \in (i,\ j)CO(Y)$ such that $f(x) \in W \subset V$. Since f is pairwise slightly semi-

continuous,by the Theorem 3.1 there is $U \subset (i, j)\ SO\ (X, x)$ such that $f((i, j)sCl(U))$ $\subset W$ and so $f(i, j)sCl(U)) \subset V$. Thus f is strongly θ-semi-continuous.

We have the following diagram:

$$
\begin{array}{lllll}
p.\delta.C & & & & \\
\Downarrow & & & & \\
P.C \quad \Rightarrow & P.a.C \quad \Rightarrow & P.\,\theta.\,C \Rightarrow & P.w.C \quad \Rightarrow & P.\,sl.C \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow \\
P.s.C \quad \Rightarrow & P.a.s.C \Rightarrow & P.s.\,\theta.C \Rightarrow & P.w.s.C \Rightarrow & P.sl.s.C \\
\Uparrow & \Uparrow & & & \\
P.st.\,\theta.s.C & \Rightarrow P.st.\,\theta.s.C & & &
\end{array}
$$

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N-POLICY FOR $M/G/1$ MACHINE REPAIR PROBLEM WITH STANDBY AND COMMON CAUSE FAILURE

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ABSTRACT

In this paper, we investigate an optimal N-policy for a single repairman machine problem with Poisson arrivals and general service time distribution. We analyze the machining system consisting of M operating machines along with S cold standby machines. For the normal operation, M machines are required; however, the system can also function with at least in $m(<M)$ machines in degraded mode. The machines may fail individually or due to comman cause. The repair times of the failed machines are independent and identically distributed random variables with general distribution. The repair rendered by the repairman is assumed to be imperfect. The governing equations are constructed by introducing the supplementary variable corresponding to remaining repair time. To obtain the steady state probabilities, recursive method is employed. A cost function is developed to calculate the operating policy at minimum cost.

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Keywords : Queue, N-policy, Cold standby, Recursive method, Supplementary variable technique, Common cause failure, Cost function.

1. Introduction. The modeling of machine repair problems has found increasing attention in recent years due to their practical applications in several areas, such as in manufacturing systems, computer systems, communication networks, etc.. For machining systems, the provosion of spare part support is common to improve the efficiency of the system but it increases the cost of the system. In many realistic situations, the server may start the service after accumulation of the pre-assigned number of the jobs. This policy seems to be cost effective. In the peresent paper, we study an optimal N-policy for $M/G/1$ machine repair problem with spares under the steady-state conditions.

In a machine repair problem, if at any time a unit fails, it is sent for repair to the repair facility. The repairman can repair only one failed unit at a time. If an

operating unit fails, it is immediately replaced by a spare unit if available. In many practical situations, the spare units are needed to operate the system continuously over a long time period. These situations are applicable in different areas such as production lines, airlines, manufacturing organizations, tele communication, etc..

Machine repair problems have been studied by several research worders in different frame-works. Some of them are **Cherian *et al.* (1987), jain *et al.* (2001),** and many more. The policy, in which the repairman will not initiate the repair of failed machines till N failed machines are accumulated to abate idle time of repairman, is termed as N-policy. Several attempts have been made to suggest control policies in this field from time to time.**Choudhury (1997)** developed a queueing system with general setup time and poisson inputes facilitated under N-policy. The queueing system with finite source and warm spares under N-policy was investigated by **Gupta (1999).**

Recently, N-policy for redundant repairable system with additional repairman was investigated by **jain (2003). Wang *et al.* (2005)** presented the maximum entropy analysis to an $M/G/1$ queueing system with unreliable server and general startup times. **Kim and Moon (2006)** discussed an $M/G/1$ queueing system under a certain service policy. A two phase batch arrival retrial queueing system with Bernaulli vacation schedule was explored by **Choudhury (2007).**

Numerous queue theorists developed machine repair problem having individual as well as common cause failures. **Hughes(1987)** considered the issue of common cause failure in machining problem. Recently, **jain et al. (2002)** investigated a flexible manufacturing system with common cause failure. **Dhillon and Li (2005)** suggested stochastic analysis of standby systems with common cause failure and human errors. **Jain and Mishra(2006)** studied multi-stage degraded machining system with common cause shock failure and state dependent rates. **Ehsani et al. (2008)** proposed a model for reliability evaluation of deregulated electric power systems with common cause failure for planning applications.

The purpose of our study is this paper is to provide a recursive method to determie the steady state probability distribution of the number of failed units in the $M/G/1$ system operating under N-policy. In section 2 we outline some assumptions and notations in order to construct the mathematical model of the system. Section 3 covers the analytical expressions if all probabilities for different states. In section 4 some important performance characteristics such as average number of failed machines in the system, the probability of repairman being idle, machine availability etc. are establised. In section 5 we determine the average length of the busy period, average length of the cycle and average length of the idle period. Also we construct the cost function in terms total expected cost per unit time. Finally conclusion is drawn in section 6.

2. **Model Description.** We consider general service system consisting of $K(M$ operating + S cold standby) machines under the supervision of single

repairman to maintain efficient operation for long run. The model is developed by considering the following assumptions:

(i) The life time of operating machines follows exponential distribution with mean $1/\lambda$.

(ii) The system may fail at any moment due to common cause with exponential failure rate λ_c.

(iii) $\alpha(0 \leq \alpha \leq 1)$ is the probability of recovering the failed machine. The renewed machines become as good as new ones.

(iv) The system will be in working state until there are m operating machines. If an operating machine fails, a cold standby machine replaces it with negligible switchover time provided it is a available at that moment.

(v) The repair times of the failed machines are independent and identically distributed (i,i,d) random variables having general distribution function.

(vi) The repairman follows N-failed machines in the system and continues the repair until he repairs all the failed machines.

For modeling purpose, we use the following notations:

N	Threshold level
B	Random variable denoting service time
K	Total number of machines, $K=M+S$.
$B(u)$	Distribution function $(d.f.)$ of B
$b(u)$	Probability density function $(p.d.f)$ of B
$U(t)$	Remaining repair time for the machine at time t
b_1	Mean repair time
α	Probability of recovering of the failed machines
$B^*(\theta)$	Laplace- Stieltijes transform (LST) of B
$B^{*(j)}(\theta)$	jth order derivative of $B^*(\theta)$ with respect to θ
λ_c	Common cause failure rate.
λ_d	Degraded failure rate, when $n \geq S+1$.

Let $Q(t)$ and $L(t)$ be the random variables denoting the status of the repairman at any instant t and the number of failed machines in the system respectivley. Define

$$Q(t) = \begin{cases} 0, & \text{if the repairman is in idle state at time } t \\ 1, & \text{if the repaiman is no working state at time } t \end{cases}$$

Using these notations, we define the following transient probabilities:

$P_{0,0}(t)$ = Prob $\{Q(t)=0, L(t)=0\}$

$P_{0,n}(t)$ = Prob $\{Q(t)=0, L(t)=0, 0 \leq n < N\}$

$P_{1,n}(t)$ = Prob $\{Q(t)=1, L(t)=n, 1 \leq n \leq K\text{-}m\}$.

For the analysis purpose, we follow the supplementary variable technique [cf. Cox, (1995)]. By introducing random variable U corresponding to remaining repair times (u) for the failed machines in repair, we shall construct the equations of different states of the system.

Let us denote

$P_{1,n}(u,t)du$ = Prob $\{Q(t)=1,\ L(t)=n,\ u<U(t)\le u+du,\ u\ge 0,\ 1\le n\le K\text{-}m\}$

$P_{1,n}(t)=\int_0^t P_{1,n}(u,t)du$.

3. The Analysis. Considering the state of the system at time t, we obtain the tansient state equations governing the model as follows :

$$\frac{d}{dt}P_{0,0}(t)=-(M\lambda+\lambda_c)P_{0,0}(t)+P_{1,1}(0,t) \qquad ...(1)$$

$$\frac{d}{dt}P_{0,n}(t)=-(M\lambda+\lambda_c)P_{0,n}(t)+M\lambda P_{0,n-1}(t),\ 1\le n\le N-1 \qquad ...(2)$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{1,1}(u,t)=-(M\lambda+\lambda_c)P_{1,1}(u,t)+P_{1,2}(0,t)b(u) \qquad ...(3)$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{1,n}(u,t)=-(M\lambda+\lambda_c)P_{1,n}(u,t)+\alpha M\lambda P_{1,n-1}(u,t)+P_{1,n+1}(0,t)b(u)$$
$$2\le n\le N-1 \qquad ...(4)$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{1,N}(u,t)=-(M\lambda+\lambda_c)P_{1,N}(u,t)+\alpha M\lambda P_{1,N-1}(u,t)+P_{1,N+1}(0,t)b(u)$$
$$+M\lambda P_{0,N-1}(u) \qquad ...(5)$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{1,n}(u,t)=-(M\lambda+\lambda_c)P_{1,n}(u,t)+\alpha M\lambda P_{1,n-1}(u,t)+P_{1,n+1}(0,t)b(u),$$
$$N+1\le n\le S \qquad ...(6)$$

$$\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial u}\right)P_{1,n}(u,t)=-((K-n)\lambda_d+\lambda_c)P_{1,n}(u,t)+\alpha(K-n+1)\lambda_d P_{1,n-1}(u,t)$$
$$+P_{1,n+1}(0,t)b(u),\qquad S+1\le n\le K-m-1 \qquad ...(7)$$

$$\frac{d}{dt}P_{1,K-m}(t)=-(m\lambda_d+\lambda_c)P_{1,K-m}(t)+\alpha(m+1)\lambda_d P_{1,K-m-1}(t) \qquad ...(8)$$

$$\frac{d}{dt}P_F(t)=\alpha(K-m)\lambda_d P_{1,K-m}(t)+\sum_{i=1}^{s}\{(1-\alpha)M\lambda+\lambda_c\}P_{1,i}(t)$$
$$+\sum_{i=S+1}^{K-m-1}\{(1-\alpha)(K-i)\lambda_d+\lambda_c\}P_{1,i}(t)+\sum_{i=0}^{N-1}\lambda_c P_{0,i}(t) \qquad ...(9)$$

For steady state probabilities, we deonote

$P_F=\lim_{t\to\infty}P_F(t);\qquad P_{0,n}=\lim_{t\to\infty}P_{0,n}(t);\qquad 1\le n\le N-1$

$$P_{1,n} = \lim_{t\to\infty} P_{1,n}(t); \qquad P_{1,n}(u) = \lim_{t\to\infty} P_{1,n}(u,t); \quad 1 \le n \le K-m$$

$$P_{0,N-1}(u) = P_{0,N-1}b(u) .$$

From equations (1)-(9), the steady state equations are obtained as

$$-(M\lambda + \lambda_c)P_{0,0} + P_{1,1}(0) = 0 , \qquad ...(10)$$

$$-(M\lambda + \lambda_c)P_{0,n} + M\lambda P_{0,n-1}(0) = 0, \qquad 1 \le n \le N-1 \qquad ...(11)$$

$$-\frac{d}{du}P_{1,1}(u) = -(M\lambda + \lambda_c)P_{1,1}(u) + P_{1,2}(0)b(u) \qquad ...(12)$$

$$-\frac{d}{du}P_{1,n}(u) = -(M\lambda + \lambda_c)P_{1,n}(u) + \alpha M\lambda P_{1,n-1}(u) + P_{1,n+1}(0)b(u), \; 2 \le n \le N-1 \qquad ...(13)$$

$$-\frac{d}{du}P_{1,N}(u) = -(M\lambda + \lambda_c)P_{1,N}(u) + \alpha M\lambda P_{1,N-1}(u) + M\lambda P_{0,N-1}(u) + P_{1,N+1}0b(u) \qquad ...(14)$$

$$-\frac{d}{du}P_{1,n}(u) = -(M\lambda + \lambda_c)P_{1,n}(u) + \alpha M\lambda P_{1,n-1}(u) + P_{1,n+1}(0)b(u), N+1 \le n \le S \qquad ...(15)$$

$$-\frac{d}{du}P_{1,n}(u) = -[(K-n)\lambda_d + \lambda_c]P_{1,n}(u) + \alpha(K-n+1)\lambda_d P_{1,n-1}(u) + P_{1,n+1}(0)b(u),$$

$$S+1 \le n \le K-m-1 \qquad ...(16)$$

$$-(m\lambda_d + \lambda_c)P_{1,K-m} + \alpha(m+1)\lambda_d P_{1,K-m-1} = 0 \qquad (17)$$

$$\alpha(K-m)\lambda_d P_{1,K-m} + \sum_{i=1}^{S}\{(1-\alpha)M\lambda + \lambda_c\}P_{1,i} + \sum_{i=S+1}^{K-m-1}\{(1-\alpha)(K-i)\lambda_d + \lambda_c\}P_{1,i}$$

$$+ \sum_{i=0}^{N-1}\lambda_c P_{0,i} = 0 \qquad ...(18)$$

Equation (10) yields

$$P_{1,1}(0) = (M\lambda + \lambda_c)P_{0,0} . \qquad ...(19)$$

Putting $n=1$, equation (11) provides

$$P_{0,1} = \frac{M\lambda}{M\lambda + \lambda_c}P_{0,0} . \qquad ...(20)$$

Now putting $n=2$, equation (11) yields

$$P_{0,2} = \left(\frac{M\lambda}{M\lambda + \lambda_c}\right)^2 P_{0,0} . \qquad ...(21)$$

Similarly by using rcursive approach, we obtain

$$P_{0,n} = \left(\frac{M\lambda}{M\lambda+\lambda_c}\right)^n P_{0,0} \qquad 1 \le n \le N-1 \ . \qquad ...(22)$$

Further let us define Laplace-Stieltjes transform as

$$B^*(\theta) = \int_0^\infty e^{-\theta u} dB(u) = \int_0^\infty e^{-\theta u} b(u) du \ . \qquad ...(23)$$

$$P_{1,n}^*(\theta) = \int_0^\infty e^{-\theta u} P_{1,n}(u) du, \qquad 1 < n \le K-m \qquad ...(24)$$

so that, we have

$$P_{1,n} = P_{1,n}^*(0) = \int_0^\infty P_{1,n}(u) du, \quad 1 \le n \le N-m \ , \qquad ...(25)$$

$$\int_0^\infty e^{-\theta u} \frac{\partial}{\partial u} P_{1,n}(u) du = \theta P_{1,n}^*(\theta) - P_{1,n}(0), \qquad ...(26)$$

$$P_{1,N-1}(u) = P_{0,N-1} b(u) \ . \qquad ...(27)$$

Taking Laplace-Stieltjes tansforms (LST) on both sides of equations (12) to (16), we have

$$(M\lambda + \lambda_c - \theta) P_{1,1}^*(\theta) = P_{1,2}(0) B^*(\theta) - P_{1,1}(0) \qquad ...(28)$$

$$(M\lambda + \lambda_c - \theta) P_{1,n}^*(\theta) = \alpha M\lambda P_{1,n-1}^*(\theta) + P_{1,n+1}(0) B^*(\theta) - P_{1,n}(0), \ 2 \le n \le N-1 \qquad ...(29)$$

$$(M\lambda + \lambda_c - \theta) P_{1,N}^*(\theta) = \alpha M\lambda P_{1,N-1}^*(\theta) + M\lambda P_{0,N-1} B^*(\theta) + P_{1,N+1}(0) B^*(\theta) - P_{1,N}(0) \qquad ...(30)$$

$$(M\lambda + \lambda_c - \theta) P_{1,n}^*(\theta) = \alpha M\lambda P_{1,n-1}^*(\theta) + P_{1,n+1}(0) B^*(\theta) - P_{1,n}(0) \ , \ N+1 \le n \le S \qquad ...(31)$$

$$[(K-n)\lambda_d + \lambda_c - \theta] P_{1,n}^*(\theta) = \alpha(K-n+1)\lambda_d P_{1,n-1}^*(\theta) + P_{1,n+1}(0) B^*(\theta) - P_{1,n}(0),$$
$$S+1 \le n \le K-m-1 \ . \qquad ...(32)$$

From equations (19) and (22), we get

$$P_{0,n} = \left(\frac{1}{M\lambda+\lambda_c}\right)\left(\frac{M\lambda}{M\lambda+\lambda_c}\right)^n P_{1,1}(0), \ 1 \le n \le N-1 \ . \qquad ...(33)$$

Using equation (33) in equation (30) and adding equations (28)-(32), we get

$$(M\lambda + \lambda_c - \theta)\sum_{n=1}^{S} P_{1,n}^*(\theta) + \sum_{n=S+1}^{K-m-1} [(K+n)\lambda_d + \lambda_c - \theta] P_{1,n}^*(\theta)$$
$$= \alpha M\lambda \sum_{n=2}^{S} P_{1,n-1}^*(\theta) + \alpha\lambda_d \sum_{n=S+1}^{K-m-1} (K-n+1) P_{1,n-1}^*(\theta)$$
$$+ \sum_{n=1}^{K-m-1} P_{1,n+1}(0) B^*(\theta) + \left[\left(\frac{M\lambda}{M\lambda+\lambda_c}\right)^N\right] P_{1,1}(0) - 1 \qquad ...(34)$$

Now using equation (33) in equation (30) and setting $\theta = M\lambda + \lambda_c$ in equations (28)-(31), we obtain,

$$P_{1,n+1}(0) = \frac{P_{1,n}(0) - \phi P_{1,1}(0) - \alpha M\lambda P^{*}_{1,n-1}(M\lambda + \lambda_c)}{B^*(M\lambda + \lambda_c)}, \qquad 1 \le n \le S \qquad ...(35)$$

where

$$\phi = \begin{cases} \left(\dfrac{M\lambda}{M\lambda + \lambda_c}\right)^n, & n = N \\ 0, & \text{otherwise} \end{cases} \quad \text{and } P^{*}_{1,n}(;) = 0 \text{ for } n < 1.$$

Now substituting $\theta = (K-n)\lambda_d + \lambda_c$ in equation (32) we get,

$$P_{1,n+1}(0) = \frac{P_{1,n}(0) - \alpha[(K-n+1)\lambda_d]P^{*}_{1,n-1}[(K-n)\lambda_d + \lambda_c]}{B^*[(K-n)\lambda_d + \lambda_c]},$$

$$S+1 \le n \le K-m-1 \cdot \qquad ...(36)$$

Differentiating equations (28)-(31) j times w.r.t. 'θ' and setting $\theta = M\lambda + \lambda_c$ we have

$$P^{*(j-1)}_{1,n}(M\lambda + \lambda_c) = -\frac{1}{j}\Big[P_{1,n+1}(0)B^{*(j)}(M\lambda + \lambda_c) + \phi P_{1,1}(0)B^{*(j)}(M\lambda + \lambda_c) +$$

$$\alpha M\lambda P^{*(j)}_{1,n-1}(M\lambda + \lambda_c)\Big], \quad 2 \le n \le S-1, \quad 1 \le n \le S-n-1 \qquad ...(37)$$

where $P^{*(0)}_{1,n}(\theta) = P^{*}_{1,n}(\theta)$ & $B^{*(j)}(\theta) = \dfrac{d^j}{d\theta^j}B^*(\theta)$ is the j^{th} derivative of $B^*(\theta)$.

Again setting $\theta = (M-r)\lambda + \lambda_c$, in equations (28)-(31), we get

$$P^{*}_{1,n}[(M-r)\lambda_d + \lambda_c] = \frac{1}{r\lambda}\Big[\alpha M\lambda P^{*}_{1,n-1}\{(M-r)\lambda + \lambda_c\} + \phi P_{1,1}(0)B^*\{(M-r)\lambda + \lambda_c\}$$

$$+ P^{*}_{1,n+1}(0)B^*\{(M-r)\lambda + \lambda_c\} - P_{1,n}(0)\Big], 1 \le n \le S, \ 1 \le r \le M-m-1. \qquad ...(38)$$

Again setting $\theta = (M-r)\lambda_d + \lambda_c$, equation (32) yields

$$P^{*}_{1,n}[(M-r)\lambda_d + \lambda_c] = \frac{1}{(S-n+r)\lambda_d}\Big[\alpha(K-n+1)\lambda_d P^{*}_{1,n-1}\{(M-r)\lambda_d + \lambda_c\} + P_{1,n+1}(0)B^*$$

$$\{(M-r)\lambda_d + \lambda_c\} - P_{1,n}(0)\Big], \ S+1 \le n \le S+r-1, \ 2 \le r \le M-m-1. \qquad ...(39)$$

Now $P_{1,2}(0), P_{1,3}(0), ..., P_{1,k-m}(0)$, can be obtained recursively with the help of equations (35)-(39) in terms of P_{00}.

Again setting $\theta = 0$ in equations (28)-(32), we have

$$P^{*}_{1,n}(0)=\frac{\alpha M\lambda P^{*}_{1,n-1}(0)+P_{1,n+1}(0)-P_{1,n}(0)}{M\lambda-\lambda_c},\quad 1\le n\le N-1 \qquad ...(40)$$

where $B^{*}(0)=1$.

Also

$$P^{*}_{1,n}(0)=\frac{\alpha M\lambda P^{*}_{1,n-1}(0)+P_{1,n+1}(0)+\phi_{1,1}(0)-P_{1,n}(0)}{M\lambda+\lambda_c},\qquad N\le n\le S \qquad ...(41)$$

and

$$P^{*}_{1,n}(0)=\frac{\alpha(K-n+1)\lambda_d P^{*}_{1,n-1}(0)+P_{1,n+1}(0)-P_{1,n}(0)}{(K-n)\lambda_d+\lambda_c},\ S+1\le n\le K-m-1 \qquad ...(42)$$

Now $P_{1,n}(0),(1\le n\le K-m)$, can be determined respectively by using equations (40)-(42) in terms of $P_{0,0}$.

From equation (4.17), we obtain

$$\therefore\ P_{1,K-m}=\frac{\alpha(m+1)\lambda_d P_{1,K-m-1}}{(m\lambda_d+\lambda_c)} \qquad ...(43)$$

The value of $P_{0,0}$ can be determined by using normalizing condition as

$$\sum_{n=0}^{N-1}P_{0,n}+\sum_{n=1}^{K-m}P^{*}_{1,n}(0)=1\ . \qquad ...(44)$$

The failure probability of the system can be obtained by

$$P_f=1-\sum_{n=0}^{N-1}P_{0,n}-\sum_{n=1}^{K-m}P^{*}_{1,n}(0) \qquad ...(45)$$

4. Some Performance Indices. In this section, we present some performance indices so that the system designer may ensure efficiency and effectiveness of the system for future design and development. Now, we formulate the performance indices in terms of queue size distribution obtained in previous section.

Let us denote the performance metrics as follows :

$E(N)$ Average number of failed machines in the system.

$E(N_q)$ Average number faild of machines that are waiting for repair in the queue.

$E(O)$ Average number of operating machines in the system.

$E(S)$ Average number of standby machines in the system.

$P(I)$ The probability of repaiman being idle.

$P(B)$ The probability of repaiman being busy.

MA Machine availability.

The above performance indices can be given in terms of queue size as follows :

(i) $$E(N)=\sum_{n=0}^{N-1}nP_{0,n}+\sum_{n=1}^{K-m}nP_{1,n} \qquad ...(46)$$

(ii) $$E(N_q) = \sum_{n=1}^{N-1}(n-1)P_{0,n} + \sum_{n=1}^{K-m}(n-1)P_{1,n} \quad ...(47)$$

(iii) $$E(O) = M - \sum_{n=S+1}^{K-m}(n-S)P_{1,n} \quad ...(48)$$

(iv) $$E(S) = \sum_{n=0}^{S}(S-n)P_{1,n} \quad ...(49)$$

(v) $$P(I) = \sum_{n=0}^{N-1} P_{0,n} = NP_{0,0} \quad ...(50)$$

(vi) $$P(B) = 1 - P(I) \quad ...(51)$$

(vii) $$MA = \frac{1}{K-m}\left[\sum_{n=0}^{N-1}(K-m-n)P_{0,n} + \sum_{n=1}^{K-m}(K-m-n)P_{1,n}\right]. \quad ...(52)$$

5. Cost Analysis. We assume that $E(B)$, $E(C)$ and $E(I)$ represent the average length of the busy period, average length of the cycle and average length of the idle period, respectively. Then average length of idle period is obtained as

$$E(I) = \frac{N}{M\lambda}. \quad ...(53)$$

The long run fraction of time for which repairman remains idle and busy, are given by

$$\frac{E(I)}{E(C)} = \sum_{n=1}^{N} nP_{0,0}. \quad ...(54)$$

and

$$\frac{E(B)}{E(C)} = 1 - \sum_{n=1}^{N} nP_{0.0}. \quad ...(55)$$

To determine the optimal value of N in the system, we construct the cost function denoting the total expected cost per unit time as given below:

$$TC(N,S) = Co\frac{E(B)}{E(C)} + C_r\frac{E(I)}{E(C)} + C_h E(N) + (C_b + C_S)\frac{1}{E(C)} + (M\lambda + \lambda_c)P^{*}_{1,K-m}(0) \quad ...(56)$$

where different cost elements used are as follows :

C_o = Cost per unit time for keeping the repairman on

C_f = Cost per unit time for keeping the repairman off.

C_h = Holding cost per unit time per machine present in the system.

C_b = Break-down cost per unit time for turning the repairman off.

C_s =Start-up cost per unit time for turning the repairman on.

6. Conclusion. In the present paper we have obtained the steady state solution for the N-policy $M/G/1$ machine repair problem with cold spares. The

concept of the common cause failure along with individual failures is incorporated, which is common in many real time machining systems. The suggested recursive method using supplementary variable technique to determine the steady-state probabilities can be easily implementable to quantify various performance indices as well as cost function. Our study provides good tradeoff for gaining initial insight of system's performance and cost incurred so as to determine the optimal service parameters. The concerned model is more sophisticated and versatile in our real life situations as it includes common cause failure and imperfect repair performed by a single repairman.

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ON GENERATING RELATIONSHIPS FOR FOX'S *H*-FUNCTION AND MULTIVARIABLE *H*-FUNCTION

By

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ABSTRACT

In this paper we establish some new results on bilinear, bilateral and multilateral generating relationship for Fox's *H*-function and multivariable *H*-function. Some known results for the Fox's *H*-function and multivariable *H*-function are also obtained as special cases of our main findings.

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1. Introduction. Chen and Shrivastava [1] gave a family of linear, bilateral and multilateral generating functions involving the sequence $\{\zeta_k^{(\lambda,\rho)}(z)\}_{k=0}^{\infty}$ defined by

$$\zeta_k^{(\lambda,\rho)}(z) = {}_uF_{\rho+v}(\alpha_1,\ldots,a_u;\Delta(\rho;1-\lambda-k),\beta_1,\ldots,\beta_v;z) \qquad \ldots(1)$$

where for convenience, $\Delta(\rho;\lambda)$ abbreviates the array of ρ parameters

$$\frac{\lambda}{\rho},\frac{\lambda+1}{\rho},\ldots,\frac{\lambda+\rho-1}{\rho} \qquad (\rho \in N = N_0 \setminus \{0\})$$

and for its multivariable extension defined by ([1], p.172, equation (5.21)).

$$Z_k^{\lambda}(\sigma_1,\ldots,\sigma_r;z_1,\ldots,z_r) = \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{A(k_1,\ldots,k_r)}{(1-\lambda-k)_K} z_1^{k_1}\ldots z_r^{k_r} \qquad \ldots(2)$$

$$(K = k_1\sigma_1 + \ldots + k_r\sigma_r; k_1 \in N_0; \lambda, \sigma_j \in C; j=1,\ldots,r)$$

where $\{A(k_1,\ldots,k_r)\}$ is a suitably bounded multiple sequence of complex numbers and $(\lambda)_k$ denotes the Pochhammer symbol.

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1, & (k=0;\lambda\neq 0) \\ \lambda(\lambda+1)\ldots(\lambda+k-1) & (k\in N;\lambda\in C) \end{cases} \qquad \ldots(3)$$

Raina[4] derived the following combinatorial identity as a special case of formula

in ([4], p. 187,equation (15)).

$$\sum_{k=0}^{\infty}\binom{\lambda+k-1}{k}\binom{\mu+k-1}{k}^{-1}\binom{\alpha+k-1}{k}{}_2F_1\left[\begin{matrix}\lambda+k,\mu-\alpha;\\ \mu+k;\end{matrix}\, z\right]z^k$$

$$=(1-z)^{-\lambda},\quad (|z|<1) \qquad \ldots(4)$$

$$\text{where } \binom{n}{k}=\frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n+k+1)} \qquad \ldots(5)$$

Recently in an earlier paper Jaimini et al. [3] generalized the results of above cited paper [1]. They proved six theorems on the generating function relationships in view of the above results (4).

The Fox's H-function defind and represented in the following manner ([2], p. 408), See also ([5],p 265, equation (1,1))

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}\{(a_p,\alpha_p)\}\\ \{(b_q,\beta_q)\}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L \phi(\xi)z^{\xi}d\xi \qquad \ldots(6)$$

where

$$\phi(\xi)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)} \qquad \ldots(7)$$

The multivariable H-function defined and represented in the following manner ([6],pp.251-252, equations (C.1)-(C.3)).

$H[z_1,\ldots,z_r]$

$$=H_{p,q:p_1,q_1;\ldots;p_r,q_r}^{0,n:m_1,n_1;\ldots;m_r,n_r}\left[\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p}:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q}:(d_j^{(1)},\delta_j^{(1)})_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right.\right]$$

$$=\frac{1}{(2\pi i)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(s_1)\ldots\phi_r(s_r)\psi(s_1,\ldots,s_r)z_1^{s_1}\ldots z_r^{s_r}ds_1\ldots ds_r; \qquad \ldots(8)$$

where $i=\sqrt{(-1)}$;

$$\phi_i(s_i)=\frac{\prod_{j=1}^{m_i}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{n_i}\Gamma(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod_{j=m_i+1}^{q_i}\Gamma(1-d_i^{(i)}+\delta_j^{(i)}s_i)\prod_{j=n_i+1}^{p_i}\Gamma(c_j^{(i)}-\gamma_j^{(i)}s_i)}\forall i\in\{1,\ldots,r\} \qquad \ldots(9)$$

$$\psi(s_1,\ldots,s_r)=\frac{\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{j=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{j=1}^{r}\alpha_j^{(i)}s_i\right)\prod_{j=1}^{q}\Gamma\left(1-b_j+\sum_{j=1}^{r}\beta_j^{(i)}s_i\right)} \quad \ldots(10)$$

In this paper some generating relations for Fox's H-function and multivariable H-function defined in (6) and (8) respectively are established by following the above cited work of Jamini et al. [3]. The importance of these results lies in the fact that they provide the extensions of the results due to Srivastava and Raina [7] and also provide a wide range of bilinear, bilateral mixed multilateral generating functions for simpler hypergeometric polynomials.

2. Main Bilateral Generating Relationship Involving Fox's *H*-Function

Result -1 . Corresponding to an identically nonvanishing function $\Omega_{\vartheta}(z_1,\ldots,z_s)$ of s complex variables $z_1,\ldots,z_s (s\in N)$ and of (complex) order ϑ, let.

$$\gamma^{(1)}_{m,\vartheta,\rho,\sigma,\lambda}[y;z_1,\ldots,z_s;t]=\sum_{k=0}^{\infty}\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)t^k}{(mk)!}H^{r,s+1}_{u+1,v}\left[y\middle|\begin{matrix}(1-\lambda-mk-\sigma mk,\varepsilon), & \{(c_u,\gamma_u)\}\\ & \{(d_v,\delta_v)\}\end{matrix}\right]$$

$$[a_k\neq 0;k\in N_0;\vartheta,\sigma\in C] \quad (11)$$

and

$$M^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}_{n,m}[y;z_1,\ldots,z_s;\eta]=\sum_{k=0}^{[n/m]}A^{\lambda,\sigma,\mu,\alpha}_{k,n,m}(y;t)\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}$$

$$\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)\eta^k}{(mk)!(n-mk)!} \quad \ldots(12)$$

where

$$A^{\lambda,\sigma,\mu,\alpha}_{k,n,m}[y;t]=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l(l)!}H^{r,s+1}_{u+1,v}\left[y\middle|\begin{matrix}(1-\lambda-n-\sigma mk,\varepsilon), & \{(c_u,\gamma_u)\}\\ & \{(d_v,\delta_v)\}\end{matrix}\right] \ldots(13)$$

then

$$\sum_{n=0}^{\infty}M^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}_{n,m}(y;z_1,\ldots,z_s;\eta)t^n$$

$$=(1-t)^{-\lambda}\gamma^{(l)}_{m,\vartheta,\rho,\sigma,\lambda}\left[\frac{y}{(1-t)^{\varepsilon}};z_1,\ldots,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m}}\right] \quad \ldots(14)$$

Result-2. Let

$$\gamma^{(2)}_{\vartheta,\rho,m}[y;z_1,\ldots,z_s;t]=\sum_{k=0}^{r}\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)t^k}{}H^{r,s}_{u,v}\left[y\left|\begin{matrix}\{(c_u,\gamma_u)\}\\\{(d_v,\delta_v)\}\end{matrix}\right.\right] \quad \ldots(15)$$

and

$$N^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}_{n,m}[y;z_1,\ldots,z_s;\eta]=\sum_{k=0}^{[n/m]}U^{\lambda,\sigma,\mu,\alpha}_{k,n,m}(y;t)\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}$$

$$\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)\eta^k}{(mk)!(n-mk)!} \quad \ldots(16)$$

where

$$U^{\lambda,\sigma,\mu,\alpha}_{k,n,m}(y;t)=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l(l)!}H^{r,s+1}_{u+1,v+1}\left[y\left|\begin{matrix}(-\lambda-n-\sigma mk-l,\varepsilon),\{(c_u,\gamma_u)\}\\\{(d_v,\delta_v)\},(-\lambda-mk-\sigma mk,\varepsilon)\end{matrix}\right.\right]. \quad \ldots(17)$$

Then

$$\sum_{n=0}^{\infty}M^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}_{n,m}(y;z_1,\ldots,z_s;\eta)t^n=(1-t)^{-(\lambda+1)}\gamma^{(2)}_{\vartheta,\rho,m}\left[\frac{y}{(1-t)^{\varepsilon}};z_1,\ldots,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m}}\right] \quad \ldots(18)$$

Result-3

Let $\gamma^{(2)}_{\vartheta,\rho,m}[y;z_1,\ldots,z_s;t]$ is defined in (15) and

$$T^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}_{n,m}[y;z_1,\ldots,z_s;\eta]=\sum_{k=0}^{[n/m]}V^{\lambda,\sigma,w,\mu,\alpha}_{k,n,m}(y;t)\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}$$

$$\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)\eta^k}{(n-mk)!} \quad \ldots(19)$$

where

$$V^{\lambda,\sigma,\omega,\mu,\alpha}_{k,n,m}(y;t)=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l(l)!}H^{r,s+1}_{u+1,v+1}\left[y\left|\begin{matrix}(-\lambda-n-\sigma mk-\omega k-l,\varepsilon),\{(c_u,\gamma_u)\}\\\{(d_v,\delta_v)\},(-\lambda-\sigma mk-(\omega+m)k,\varepsilon)\end{matrix}\right.\right] \quad \ldots(20)$$

then

$$\sum_{n=0}^{\infty}T^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}_{n,m}[y;z_1,\ldots,z_s;\eta]t^n=(1-t)^{-(\lambda+1)}\gamma^{(2)}_{\vartheta,\rho,m}\left[\frac{y}{(1-t)^{\varepsilon}};z_1,\ldots,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m+\omega}}\right] \quad \ldots(21)$$

Result-4

Let

$$\gamma^{(\cdot,\cdot)}_{m,\sigma,\vartheta,\rho,\omega,\lambda}[y;z_1,\ldots,z_s;t]=\sum_{k=0}^{\infty}(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)t^k$$

$$H^{r,s+1}_{u+1,\upsilon+1}\left[y \middle| \begin{matrix} (1-\lambda-mk-\sigma mk-\omega k,\varepsilon),\{(c_u,\gamma_u)\} \\ \{(d_\upsilon,\delta_\upsilon)\},(-\lambda-\sigma mk-(\omega+m)k,\varepsilon) \end{matrix} \right] \qquad ...(22)$$

and .

$$\theta^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}_{n,m}\left[y;z_1,\ldots,z_s;\eta\right] = \sum_{k=0}^{[n/m]} W^{\lambda,\sigma,\omega,\mu,\alpha}_{k,n,m}(y;t)$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)\eta^k}{(n-mk)!} \qquad ...(23)$$

where

$$W^{\lambda,\sigma,\omega,\mu,\alpha}_{k,n,m}(y;t) = \sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l (l)!}H^{r,s+1}_{u+1,\upsilon+1}\left[y \middle| \begin{matrix} (1-\lambda-n-\sigma mk-\omega k-l,\varepsilon),\{(c_u,\gamma_u)\} \\ \{(d_\upsilon,\delta_\upsilon)\},(-\lambda-\sigma mk-(\omega+m)k,\varepsilon) \end{matrix} \right] (24)$$

Then

$$\sum_{n=0}^{\infty}\theta^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}_{n,m}\left[y;z_1,\ldots,z_s;\eta\right] = (1-t)^{-\lambda}\gamma^{(4)}_{m,\sigma,\vartheta,\rho,\omega,\lambda}\left[\frac{y}{(1-t)^{\varepsilon}};z_1,\ldots,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m+\omega}}\right] \qquad ...(25)$$

Proof of Result-1

We denote the left hand side of the assertion (14) of Result-1 by $H[x,y,t]$ then we use to the definitons in (12) and (13) we have :

$$H[x,y,t] = \sum_{n,l=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l (l)!}\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}$$

$$\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)\eta^k}{(mk)!(n-mk)!}H^{r,s+1}_{u+1,\upsilon}\left[y \middle| \begin{matrix} (1-\lambda-n-\sigma mk-l,\varepsilon),\{(c_u,\gamma_u)\} \\ \{(d_\upsilon,\delta_\upsilon)\} \end{matrix} \right]t^n .$$

Now using the definition of Fox's H-function from(6) and changing the order of summation and integraiton and then on making series rearrangement therein, it takes the following from:

$$H[x,y,t] = \frac{1}{2\pi i}\int_L \phi(\xi)y^{\xi}\left[\sum_{n,l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+mk+\sigma mk)_l (l)!}\binom{\mu+n+mk+\sigma mk-1}{n}^{-1}\right.$$

$$\left.\binom{\alpha+n+mk+\sigma mk-1}{n}\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)}{(mk)!(n)!}\Gamma(\lambda+n+mk+\sigma mk+l+\varepsilon\xi)\eta^k t^{n+mk}\right]d\xi$$

Now in view of the relation

$$\frac{\Gamma(\rho+n+l)}{n!}=(\rho+n)_l\binom{\rho+n-1}{n}\Gamma(\rho) \qquad ...(26)$$

and then interpreting the inner series into Gauss' hypergeometric function ${}_2F_1$ we have:

$$H[x,y,t]=\frac{1}{2\pi i}\int_L \phi(\xi)y^{\xi}\left[\sum_{k=0}^{\infty}\left\{\sum_{n=0}^{\infty}\binom{\lambda+n+mk+\sigma mk+\varepsilon\xi-1}{n}\binom{\mu+n+mk+\sigma mk-1}{n}^{-1}\right.\right.$$

$$\left.\binom{\alpha+n+mk+\sigma mk-1}{n}{}_2F_1\left[\begin{matrix}\lambda+n+mk+\sigma mk+\varepsilon\xi,\mu-\alpha;\\ \mu+n+mk+\sigma mk;\end{matrix}t\right]t^n\right\}$$

$$\left.\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)\eta^k t^{mk}}{(mk)!}\Gamma(\lambda+mk+\sigma mk+\varepsilon\xi)\right]d\xi.$$

Now using the combinatorial identity (4) and then on interpreting the resulting contour into H-function with the helf of (6), we atonce arrive at the desired result in (14).

Similarly the proof of Results-2,3,4,would run parallel to that of Result-1, which we have already detailed above fairly adequately.

3. Some generating relationship involving H-function of seceral variables. The Results-5,6,7,8 given below are established for the multivariable H-function defined in (8) by following the corresponding result proved in section-2.

Result-5

Let $$\gamma^{(5)}_{\vartheta,\rho,m,\sigma,\lambda}[y_1,\ldots,y_r;z_1,\ldots,z_s;t]=\sum_{k=0}^{\infty}\frac{a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)t^k}{(mk)!}H^{0,\upsilon+1:u_1,\upsilon_1;\ldots;u_r,\upsilon_r}_{p+1,q:p_1,q_1;\ldots;p_r,q_r}\left[\begin{matrix}y_1(1-t)^{-\varepsilon_1}\\ \vdots\\ y_r(1-t)^{-\varepsilon_r}\end{matrix}\right.$$

$$\left|\begin{matrix}(1-\lambda-\sigma mk-mk;\varepsilon_1,\ldots,\varepsilon_r),(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)})_{1,p}:(c_j^{(1)},\gamma_j^{(1)})_{1,p_1},\ldots,(c_j^{(r)},\gamma_j^{(r)})_{1,p_r}\\ (b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q}:(d_j^{(1)},\delta_j^{(1)})_{1,q_1},\ldots,(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right]$$

and ...(27)

$$R^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}_{n,m}[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta]=\sum_{k=0}^{[n/m]}B^{\lambda,\sigma,\mu,\alpha}_{k,n,m}[y_1,\ldots,y_r;z_1,\ldots,z_s;t]\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}$$

$$\begin{pmatrix} \alpha + n + \sigma mk - 1 \\ n - mk \end{pmatrix} \frac{a_k \Omega_{\vartheta+\rho k}(z_1, \ldots, z_s)}{(mk)!\,(n-mk)!} \eta^k \;, \qquad \ldots(28)$$

where

$$B_{k,n,m}^{\lambda,\sigma,\mu,\alpha}(y_1, \ldots, y_r; t) = \sum_{l=0}^{\infty} \frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l (l)!} H^{0,\upsilon+1:u_1,v_1;\ldots;u_r,v_r}_{p+1,q:p_1,q_1;\ldots;p_r,q_r} \left[\begin{matrix} y_1 \\ \vdots \\ y_r \end{matrix} \middle| \begin{matrix} (1-\lambda-n-\sigma mk-l;\varepsilon_1,\ldots,\varepsilon_r), \end{matrix} \right.$$

$$\left. \begin{matrix} \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p} : \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q} : \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{matrix} \right] \qquad \ldots(29)$$

Then

$$\sum_{n=0}^{\infty} R_{n,m}^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta]t^n$$

$$= (1-t)^{-\lambda} \gamma_{\vartheta,\rho,m,\sigma,\lambda} \left[\frac{y_1}{(1-t)^{\varepsilon_1}}, \ldots, \frac{y_r}{(1-t)^{\varepsilon_r}}; z_1, \ldots, z_s; \frac{\eta t^m}{(1-t)^{(\sigma+1)m}} \right] \qquad \ldots(30)$$

Result-6

Let $\gamma_{m,\vartheta,\rho}^{(6)}[y_1,\ldots,y_r;z_1,\ldots,z_s;t] = \sum_{k=0}^{\infty} (-1)^{mk} \alpha_k \Omega_{\vartheta+\rho k}(z_1,\ldots,z_s) t^k H^{0,\upsilon:u_1,v_1;\ldots;u_r,v_r}_{p,q:p_1,q_1;\ldots;p_r,q_r}$

$$\left[\begin{matrix} y_1 \\ \vdots \\ y_r \end{matrix} \middle| \begin{matrix} \left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,p} : \left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1},\ldots,\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \\ \left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q} : \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1},\ldots,\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \end{matrix} \right] \qquad \ldots(31)$$

and

$$S_{n,m}^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta] = \sum_{k=0}^{[n/m]} E_{k,n,m}^{\lambda,\sigma,\mu,\alpha}[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta] \begin{pmatrix} \mu+n+\sigma mk-1 \\ n-mk \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \alpha+n+\sigma mk-1 \\ n-mk \end{pmatrix} \frac{(-1)^{mk} a_k \Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)}{(n-mk)!} \eta^k \qquad \ldots(32)$$

where

$$E_{k,n,m}^{\lambda,\sigma,\mu,\alpha}(y_1,\ldots,y_r;t) = \sum_{l=0}^{\infty} \frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l (l)!} H^{0,\upsilon+1:u_1,v_1;\ldots;u_r,v_r}_{p+1,q+1:p_1,q_1;\ldots;p_r,q_r} \left[\begin{matrix} y_1 \\ \vdots \\ y_r \end{matrix} \middle| \begin{matrix} (-\lambda-n-\sigma mk-l;\varepsilon_1,\ldots,\varepsilon_r), \\ (-\lambda-\sigma mk-mk;\varepsilon_1,\ldots,\varepsilon_r), \end{matrix} \right.$$

$$\left.\begin{matrix}(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)})_{1,p} : (c_j^{(1)},\gamma_j^{(1)})_{1,p_1};...;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (b_j;\beta_j^{(1)},...,\beta_j^{(r)})_{1,q} : (d_j^{(1)},\delta_j^{(1)})_{1,q_1};...;(d_j^{(r)},\delta_j^{(r)})_{1,q_r}\end{matrix}\right], \quad ...(33)$$

Then

$$\sum_{n=0}^{\infty} S_{n,m}^{\vartheta,\rho,\lambda,\sigma,\mu,\alpha}[y_1,...,y_r;z_1,...,z_r;\eta]t^n$$

$$= (1-t)^{-(\lambda+1)}\gamma_{m,\vartheta,\rho}^{(b)}\left[\frac{y_1}{(1-t)^{\varepsilon_1}}\cdots\frac{y_r}{(1-t)^{\varepsilon_r}};z_1,...,z_r;\frac{\eta t^m}{(1-t)^{(\sigma+1)/n}}\right] \quad ...(34)$$

Result-7. Let $\gamma_{m,\vartheta,\rho}^{(6)}[y_1,...,y_r;z_1,...,z_r;t]$ is defined in (31)

and

$$U_{n,m}^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}[y_1,...,y_r;z_1,...,z_s;\eta] = \sum_{k=0}^{[n/m]} F_{k,n,m}^{\lambda,\sigma,\omega,\mu,\alpha}[y_1,...,y_r:t]$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\mu+n+\sigma mk-1}{n-mk}\frac{(-1)^{mk}a_k\Omega_{\vartheta+\rho k}(z_1,...,z_s)}{(n-mk)!}\eta^k \quad ...(35)$$

where

$$F_{k,n,m}^{\lambda,\sigma,\omega,\mu,\alpha}[y_1,...,y_r;t] = \sum_{l=0}^{r}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l(l)!}$$

$$H_{p+1,q+1:p_1,q_1,...,p_r,q_r}^{o,v+1:u_1,v_1,...,u_r,v_r}\left[\begin{matrix}y_1\\ \vdots\\ y_r\end{matrix}\left|\begin{matrix}(-\lambda-n-\sigma mk-\omega k-l;\varepsilon_1,...\varepsilon_r),(a_j;\alpha_j^{(1)},...,\alpha_j^{(r)})_{1,p}\\ \\ (-\lambda-wk-\sigma mk-mk;\varepsilon_1,...\varepsilon_r),(b_j;\beta_j^{(1)},...,\beta_j^{(r)})_{1,p}\end{matrix}\right.\right] \quad ...(36)$$

Then

$$\sum_{n=0}^{\infty} U_{n,m}^{\vartheta,\rho,\lambda,\omega,\mu,\alpha}[y_1,...,y_r;z_1,...,z_s;\eta]t^n$$

$$= (1-t)^{-(\lambda+1)}\gamma_{m,\vartheta,p}^{(6)}\left[\frac{y_1}{(1-t)^{\varepsilon_1}},...,\frac{y_r}{(1-t)^{\varepsilon_r}};z_1,...,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m+\omega}}\right]. \quad ...(37)$$

Result-8. Let $\gamma_{m,\vartheta,p,\lambda,\sigma,\alpha}^{(8)}[y_1,...,y_r;z_1,...,z_s;t] = \sum_{k=0}^{\infty}(-1)^{mk}a_k\Omega_{\vartheta+pk}(z_1,...,z_s)t^k$

$$H^{0,\upsilon+1;u_1,\ldots,u_r,\upsilon,}_{p+1,q+1;p_1,q_1,\ldots,p_r,q_r}\left[\begin{matrix}y_1\\ \vdots\\ y_r\end{matrix}\middle|\begin{matrix}(1-\lambda-mk-\omega k-\sigma mk;\varepsilon_1,\ldots,\varepsilon_r),a_j;\alpha_{,\ldots,}^{(\,)}\alpha_j^{(r)})_{1,p}:\\ (-\lambda-mk-\omega k-\sigma mk;\varepsilon_1,\ldots,\varepsilon_r),\left(b_j;\ldots,\beta_j^{(r)}\right)_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{r}\right)_{1,p_r}\\ \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right] \qquad ...(38)$$

and

$$V_{n,m}^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}\left[y_1,\ldots,y_r;z_1,\ldots,z_s;\eta\right]=\sum_{k=0}^{[n/m]}G_{k,n,m}^{\lambda,\sigma,\omega,\mu,\alpha}\left[y_1,\ldots,y_r;t\right]$$

$$\binom{\mu+n+\sigma mk-1}{n-mk}^{-1}\binom{\alpha+n+\sigma mk-1}{n-mk}(-1)^{mk}\,a_k\Omega_{\vartheta+\rho k}(z_1,\ldots,z_s)\eta^k \qquad ...(39)$$

where

$$G_{k,n,m}^{\lambda,\sigma,\omega,\mu,\alpha}\left[y_1,\ldots,y_r;t\right]=\sum_{l=0}^{\infty}\frac{(\mu-\alpha)_l t^l}{(\mu+n+\sigma mk)_l(l)!}$$

$$H^{0,\upsilon+1;u_1,\upsilon_1,\ldots,u_r,\upsilon_r}_{p+1,q+1;p_1,q_1,\ldots,p_r,q_r}\left[\begin{matrix}y_1\\ \vdots\\ y_r\end{matrix}\middle|\begin{matrix}(1-\lambda-n-\omega k-\sigma mk-l;\varepsilon_1,\ldots,\varepsilon_r),(a_j;\alpha_j^{(1)},\alpha_j^{(r)})_{1,p}:\\ (-\lambda-\sigma mk-\omega k-mk;\varepsilon_1,\ldots,\varepsilon_r),(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)})_{1,q}:\end{matrix}\right.$$

$$\left.\begin{matrix}\left(c_j^{(1)},\gamma_j^{1}\right)_{1,p_1},\ldots,\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r}\\ \left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1},\ldots,\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}\end{matrix}\right]. \qquad ...(40)$$

Then

$$\sum_{n=0}^{\infty}V_{n,m}^{\vartheta,\rho,\lambda,\sigma,\omega,\mu,\alpha}\left[y_1,\ldots,y_r;z_1,\ldots,z_s:\eta\right]t^n$$

$$=(1-t)^{-\lambda}\gamma^{(8)}_{m,\vartheta,\rho,\lambda,\sigma,\omega}\left[\frac{y_1}{(1-t)^{\varepsilon_1}},\ldots,\frac{y_r}{(1-t)^{\varepsilon_r}};z_1,\ldots,z_s;\frac{\eta t^m}{(1-t)^{(\sigma+1)m+\omega}}\right]. \qquad ...(41)$$

4. Special Cases. If Results- 1 to 5 and in Result- 8 we take $\Omega_{\vartheta+pk}(z_1,\ldots,z_r)\to 1,\sigma=0$ and $\mu=\alpha$ these results reduce to the respective known result in ([7],pp.37-44, equations (1.10), (1.14), (3.3), (5.3), (6.9), (6.6) at $\beta=0$).

If in the result of sections-2, 3 we take $\sigma=0$, and $\Omega_{\vartheta+pk}(z_1,\ldots,z_s)\to 1$ then

these result are reduced into certain families of new generating functions associated with the Fox's H-function and multivariable H-function, but we skip the results here.

All the results of sections 2 and 3, the product of the essentially arbitary cofficients

$$a_k \neq 0 (k \in N_0)$$

and the identically nonvanishing function

$$\Omega_{\vartheta+\rho k}(z_1,...,z_s)(k \in N_0; \rho, s \in N; \vartheta \in C)$$

can indeed be notationally into one set of essentially arbitrary (and identically nonvanishing) coefficients depending on the order ϑ and one, two or more variables. In view to applying such results as in section 2 above to derive bilateral generating relationship involving Fox's H-function and as in section 3 to derive mixed multilateral generating relationships involving multivariable H-function. We find it to be convenient to specialize a_k and $\Omega_k(z_1,...,z_s)$ individually as well as separately. Our general results asserted by sections 2 and 3 can be shown to yield various families of bilateral and mixed multilateral generating relations for the specific function generated in these families but there are not recorded due to lack of space.

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ON DECOMPOSITIONS OF PROJECTIVE CURVATURE TENSOR IN CONFORMAL FINSLER SPACE

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ABSTRACT

M.S. Knebelman [1] has developed conformal geometry or generalised metric spaces. The projective tensor and curvature tensors in conformal Finisler space were discusssed by Mishra ([2][3]). The decomposition of recurrent curvature tensor in an areal space of submetric class were discussed by M. Gamma [6]. The decomposition of recurrent curvature tensor in Finsler manifold was studied by Sinha and Singh [5]. Singh and Gatoto [9] have also studied the decomposition of curvature tensor in recurrent conformal Finsler space. The purpose of the present paper is to decompose the Projective curvature tensor and study the identities satisfied by projective curvature tensor in conformal Finsler space.

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1. Introduction. Let us considered two distinct metric functions $F(x,\dot{x})$ and $\overline{F}(x,\dot{x})$ be defined over an n-dimentional space F_n, both of which satiesfy the requisite conditions for a Finsler space. Th ecorresponding two metric tensor and $\overline{g}^{ij}(x,\dot{x})$ and $\overline{g}^{ij}(x,\dot{x})$ resulting from these functions are called conformal. If there exists a factor of proportionality between two metric tensors, Knebelman has proved that the factor of proportionality between them is at most a point function. Thus we have

$$\overline{g}^{ij}(x,\dot{x}) = e^{-2\sigma} g^{ij}(x,\dot{x}) \qquad ...(1)$$

$$\overline{g}^{ij}(x,\dot{x}) = e^{-2\sigma} g^{ij}(x,\dot{x}) \qquad ...(2)$$

where

$$\sigma = \sigma(x) \qquad ...(3)$$

$$F(x,\dot{x}) = e^{\sigma} F(x,\dot{x}) \qquad ...(4)$$

The space equiped with quantities $\overline{F}(x,\dot{x})$ and $\overline{g}(x,\dot{x})$ etc is called a conformal Finsler space, it is denoted by $\overline{F}_n$.

$$B^{ij}(x,\dot{x}) = \frac{1}{2}F^2 g^{ij} - \dot{x}^i \dot{x}^j, \qquad \dots (5)$$

where B^{ij} are homogeneous of the second degree in there directional arguements.

The following geometric entities of the conformal Finsler space are given by [7] and [8].

$$\overline{G}^i(x,\dot{x}) = G^i(x,\dot{x}) - \sigma_m B^{im}(x,\dot{x}) \qquad \dots(6)$$

$$\overline{G}^i_{jk}(x,\dot{x}) = G^i_{jk}(x,\dot{x}) - \dot{\partial}_k \dot{\partial}_j B^{im}(x,\dot{x})\sigma_m, \qquad \dots(7)$$

$$\overline{G}^i_{jkh}(x,\dot{x}) = G^i_{jkh}(x,\dot{x}) - \dot{\partial}_h \dot{\partial}_k \dot{\partial}_j B^{im}(x,\dot{x})\sigma_m \qquad \dots(8)$$

$$B^{ij}(x,\dot{x}) = \frac{1}{2}F^2 g^{ij} - \dot{x}^i \dot{x}^j \ , \qquad \dots(9)$$

where $G^i_{jkh}(x,\dot{x})$ are the Berwald's connection coefficients. They satisfy

$$\dot{\partial}_j G^i_k(x,\dot{x}) = G^i_{jk}. \qquad \dots(10)$$

and the functions B^{ij} are homogeneous of the second degree in three direcitonal arguments.

2. Identities Satisfied by the Conformally Changed Projective Curvature Tensor. The tensor W^i_h and W^i_{kh} transform under the conformal change as follow [2].

$$\overline{W}^i_h = W^i_h - \sigma_m\Big[2B^{im}_h - \left(\dot{\partial}_h B^{im}\right)_{(r)}\dot{x}^r - \frac{1}{n-1}\delta^i_h\left\{2B^{pm}_{(p)} - \left(\dot{\partial}_p B^{pm}\right)_{(r)}\dot{x}^r\right\} - \frac{\dot{x}^r}{n^2-1}$$

$$\left\{(2n-1)\left(\dot{\partial}_p B^{pm}\right)_{(h)} - (n+1)\left(\dot{\partial}_h B^{pm}\right)_{(p)} + 2(n-2)B^{rm}G^p_{rph} - (n-2)\dot{x}^r\left(\dot{\partial}_p \dot{\partial}_h B^{pm}\right)_{(r)}\right\}\Big]$$

$$+\sigma_{m(r)}\dot{x}^r\left\{\dot{\partial}_h B^{im} - \frac{1}{n-1}\delta^i_h \dot{\partial}_p B^{pm} - \frac{n-2}{n^2-1}\dot{x}^i \dot{\partial}_h \dot{\partial}_p B^{pm}\right\} - \sigma_{m(h)}\Big\{2B^{im} - \frac{2n-1}{n^2-1}$$

$$\dot{x}^i \dot{\partial}_p B^{pm} + \sigma_{m(p)}\left\{\frac{2}{n-1}\delta^i_h B^{pm} - \frac{\dot{x}^2}{n-1}\dot{\partial}_h B^{pm}\right\} + \sigma_m \sigma_r\Big[2B^{sm}\dot{\partial}_h \dot{\partial}_s B^{ir} - \left(\dot{\partial}_h B^{sm}\right)\dot{\partial}_s B^{ir}$$

$$-\frac{1}{n-1}\delta^i_h\left\{2B^{sm}\dot\partial_p\dot\partial_s B^{pr}-\left(\dot\partial_p B^{sm}\right)\dot\partial_s B^{pr}\right\}+\frac{2\dot x^s}{n^2-1}\left\{(n+1)\left(\dot\partial_{|p}B^{sm}\right)\dot\partial_{h|}\dot\partial_s B^{pr}\right.$$

$$\left.+(n-2)B^{sm}\dot\partial_p\dot\partial_h\dot\partial_s B^{pr}\right\}\Big], \qquad \text{...(11)}$$

$$\overline{W}^i_{kh}=W^i_{kh}-2\sigma_m\left[\left(\dot\partial_{[k}B^{im}\right)_{(h)]}-\frac{\dot x^i}{n+1}\left\{\left(\dot\partial_p\dot\partial_{[k}B^{pm}\right)_{(h)]}+\left(\dot\partial_{[k}B^{rm}\right)G^p_{h]rp}\right\}+\frac{1}{n^2-1}\delta^i_{[k}\left\{(n+1)\right.\right.$$

$$\left.\left.\left(\dot\partial_{h]}B^{pm}\right)_{(p)}-n\left(\dot\partial_p B^{pm}\right)_{(h)]}-\left(\dot\partial_{h]}\dot\partial_p B^{pm}\right)_{(r)}\dot x^r+2B^{rm}G^p_{h]rp}\right\}\right]+2\sigma_{m|(k)}\left\{\dot\partial_{h]}B^{im}-\frac{n}{n^2-1}\right.$$

$$\left.\delta^i_{h]}\dot\partial_p B^{pm}-\frac{\dot x^s}{n+1}\dot\partial_{h]}\dot\partial_p B^{pm}\right\}+\frac{2}{n^2-1}\sigma_{m(p)}\delta^i_{[k}\left\{\dot x^p\dot\partial_{h]}\dot\partial_r B^{rm}-\right.$$

$$\left.(n+1)\dot\partial_{h]}B^{pm}\right\}+2\sigma_m\sigma_r\left[\left(\dot\partial_{[k}B^{sm}\left\{\dot\partial_{h]}\dot\partial_s B^{ir}-\frac{\dot x^s}{n+1}\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}\right\}\frac{1}{n-1}\delta^i_{[k}\left\{\left(\dot\partial_{h]}B^{sm}\right)\right.\right.\right.$$

$$\left.\left.\dot\partial_p\dot\partial_s B^{pr}-\left(\dot\partial_{h]}\dot\partial_s B^{pr}\right)\dot\partial_p B^{sm}+\frac{2}{n-1}B^{sm}\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}\right\}\right] \qquad \text{...(12)}$$

R.B. Mishra [2] have introduced to obtain the conformal transformation or projective curvature tensor W^i_{jkh} by differentiating (12) with respect to $\dot x^j$.

$$\overline{W}^i_{jkh}=W^i_{jkh}+2\sigma_m\left[\left(\dot\partial_{[k}B^{ir}\right)G^m_{h]jr}-\dot\partial_j\left(\dot\partial_{[k}B^{im}\right)_{(h)]}+\frac{\dot x^s}{n+1}\left\{\dot\partial_j\dot\partial_{[k}\left(\dot\partial_p B^{pm}\right)_{(h)]}\right\}+\frac{\delta^s_j}{n+1}\right.$$

$$\left\{\dot\partial_p\left(\dot\partial_{[k}B^{pm}\right)_{(h)]}-\left(\dot\partial_{[k}B^{pr}\right)G^m_{h]pr}\right\}-\frac{\delta^i_{[k}}{n^2-1}\left\{n\dot\partial_j\left(\dot\partial_{h]}B^{pm}\right)_{(p)}-n\dot\partial_j\left(\dot\partial_p B^{pm}\right)_{(h)]}\right.$$

$$\left.\left.+\dot\partial_{h]}\left(\dot\partial_j B^{pm}\right)_{(p)]}-\dot\partial_{h]}\left(\dot\partial_p B^{pm}\right)_{(j)]}\right\}-\frac{\dot x^i\delta^i_{[k}}{n^2-1}\dot\partial_j\left\{\dot\partial_{h]}\left(\dot\partial_l B^{pm}\right)_{(p)}-\dot\partial_{h]}\left(\dot\partial_p B^{pm}\right)_{(l)}\right\}\right]+2\sigma_{m|[k}\left\{\dot\partial_{h]}\left(\dot\partial_j B^{im}\right)\right.$$

$$\left.-\frac{\dot x^i}{n+1}\dot\partial_j\left(\dot\partial_{h]}\dot\partial_p B^{pm}\right)-\frac{\delta^i_j}{n+1}\dot\partial_{h]}\left(\dot\partial_p B^{pm}\right)\right\}+\frac{2n\,\delta^i_{[k}}{n^2-1}\left\{\sigma_{m(h)]}\left(\dot\partial_j\dot\partial_p B^{pm}\right)-\sigma_{m(p)}\left(\dot\partial_j\dot\partial_{h]}B^{pm}\right)\right\}$$

$$+2\sigma_m\sigma_r\left[\dot\partial_j\left(\dot\partial_{[k}B^{sm}\right)\dot\partial_{h]}\dot\partial_s B^{ir}-\frac{\dot x^i}{n+1}\left\{\dot\partial_j\dot\partial_{[k}\left(\dot\partial_p B^{sm}\right)\dot\partial_{h]}\dot\partial_s B^{pr}+\dot\partial_{[k}\left(\dot\partial_p B^{sm}\right)\dot\partial_j\left(\dot\partial_{h]}\dot\partial_s B^{pr}\right)\right.\right.$$

$$\left.+\dot\partial_j\left(\dot\partial_{[k}B^{sm}\right)\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}+\left(\dot\partial_{[k}B^{sm}\right)\dot\partial_j\dot\partial_{h]}\left(\dot\partial_p\dot\partial_s B^{pr}\right)\right\}-\frac{\delta^i_j}{n+1}\left\{\dot\partial_p\left(\dot\partial_{[h}B^{sm}\right)\dot\partial_{k]}\dot\partial_s B^{pr}\right\}$$

$$+\frac{n\delta^i_{[k}}{n^2-1}\{(\dot\partial_j\dot\partial_{h]}B^{sm})\dot\partial_p\dot\partial_s B^{pr}-(\dot\partial_j\dot\partial_p B^{sm})\dot\partial_{h]}\dot\partial_s B^{pr}+(\dot\partial_{h]}B^{sm})\dot\partial_j\dot\partial_p\dot\partial_s B^{pr}$$

$$-(\dot\partial_p B^{sm})\dot\partial_j\dot\partial_{h]}\dot\partial_s B^{pr}\}+\frac{\delta^i_{[k}}{n^2-1}\{\dot\partial_{h]}(\dot\partial_j B^{sm})\dot\partial_p\dot\partial_s B^{pr}-\dot\partial_{h]}(\dot\partial_p B^{sm})\dot\partial_j\dot\partial_s B^{pr}$$

$$+(\dot\partial_j B^{sm})\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}-(\dot\partial_p B^{sm})\dot\partial_{[h}\dot\partial_j\dot\partial_s B^{pr}\}+\frac{\dot x^l\delta^i_{[k}}{n^2-1}\Big[\dot\partial_j(\dot\partial_{h]}\dot\partial_l B^{sm})\dot\partial_p\dot\partial_s B^{pr}$$

$$-\dot\partial_j\dot\partial_{h]}(\dot\partial_p B^{sm})\dot\partial_l\dot\partial_s B^{pr}+\dot\partial_{h]}(\dot\partial_l B^{sm})\dot\partial_j\dot\partial_p\dot\partial_s B^{pr}-\dot\partial_{h]}(\dot\partial_p B^{sm})\dot\partial_j\dot\partial_l\dot\partial_s B^{pr}$$

$$+\dot\partial_j(\dot\partial_l B^{sm})\dot\partial_{h]}(\dot\partial_p\dot\partial_s B^{pr})-\dot\partial_j(\dot\partial_p B^{sm})\dot\partial_{h]}(\dot\partial_l\dot\partial_s B^{pr})+(\dot\partial_l B^{sm})\dot\partial_j\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}$$

$$-(\dot\partial_p B^{sm})\dot\partial_j\dot\partial_{h]}\dot\partial_l\dot\partial_s B^{pr}\}\Big]+\frac{2\delta^i_{[k}}{n^2-1}\{\sigma_{m(j)}\dot\partial_{h]}(\dot\partial_p B^{pm})-\sigma_{m(p)}\dot\partial_{h]}(\dot\partial_j B^{pm})\}$$

$$+\frac{2\dot x^l\delta^i_{[k}}{n^2-1}\{\sigma_{m(l)}\dot\partial_j(\dot\partial_{h]}\dot\partial_p B^{pm})-\sigma_{m(p)}\dot\partial_j\dot\partial_{h]}(\dot\partial_l B^{pm})\}. \tag{13}$$

Multiply (13) by $\bar g_{iu}$, we get

$$\overline W_{jkh}\bar g_{iu}=e^{2\sigma}g_{iu}W_{jkh}+2e^{2\sigma}\sigma_m g_{iu}\Big[\dot\partial_{[k}B^{ir})G^m_{h]jr}-\dot\partial_j(\dot\partial_{[k}B^{im})_{(h)]}+\frac{\dot x^i}{n+1}\{\dot\partial_j\dot\partial_{[k}(\dot\partial_p B^{pm})_{(h)]}\}$$

$$+\frac{\delta^i_j}{n+1}\{\dot\partial_p(\dot\partial_{[k}B^{pm})_{(h)]}-(\dot\partial_{[k}B^{pr})G^m_{h]pr}\}-\frac{\delta^i_{[k}}{n^2-1}\{n\dot\partial_j(\dot\partial_{h]}B^{pm})_{(p)}-n\dot\partial_j(\dot\partial_p B^{pm})_{(h)]}$$

$$+\dot\partial_{h]}(\dot\partial_j B^{pm})_{(p)]}-\dot\partial_{h]}(\dot\partial_p B^{pm})_{(j)]}\}-\frac{\dot x^i\delta^i_{[k}}{n^2-1}\dot\partial_j\{\dot\partial_{h]}(\dot\partial_l B^{pm})_{(p)}-\dot\partial_{h]}(\dot\partial_p B^{pm})_{(l)}\}\Big]$$

$$+2e^{2\sigma}g_{iu}\Big[\sigma_{m[k}\Big\{\dot\partial_{h]}(\dot\partial_j B^{im})-\frac{\dot x^i}{n+1}\dot\partial_j(\dot\partial_{h]}\dot\partial_p B^{pm})-\frac{\delta^i_j}{n+1}\dot\partial_{h]}(\dot\partial_p B^{pm})\Big\}$$

$$+\frac{n\delta^i_{[k}}{n^2-1}\{\sigma_{m(h)]}(\dot\partial_j\dot\partial_p B^{pm})-\sigma_{m(p)}(\dot\partial_j\dot\partial_{h]}B^{pm})\}\Big]+2e^{2\sigma}g_{iu}\sigma_m\sigma_r\Big[\dot\partial_j(\dot\partial_{[k}B^{sm})\dot\partial_{h]}\dot\partial_s B^{ir}$$

$$-\frac{\dot x^l}{n+1}\{\dot\partial_j\dot\partial_{[k}(\dot\partial_p B^{sm})\dot\partial_{h]}\dot\partial_s B^{pr}+\dot\partial_{[k}(\dot\partial_p B^{sm})\dot\partial_j(\dot\partial_{h]}\dot\partial_s B^{pr})+\dot\partial_j(\dot\partial_{[k}B^{sm})\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}$$

$$+(\dot\partial_{[k}B^{sm})\dot\partial_j\dot\partial_{h]}(\dot\partial_p\dot\partial_s B^{pr})\}-\frac{\delta^i_j}{n+1}\{\dot\partial_p(\dot\partial_{[h}B^{sm})\dot\partial_{k]}\dot\partial_s B^{pr}\}+\frac{n\delta^i_{[k}}{n^2-1}\{(\dot\partial_j\dot\partial_{h]}B^{sm})\dot\partial_p\dot\partial_s B^{pr}$$

$$-(\dot\partial_j\dot\partial_p B^{sm})\dot\partial_{h]}\dot\partial_s B^{pr}+(\dot\partial_{h]}B^{sm})\dot\partial_j\dot\partial_p\dot\partial_s B^{pr}-(\dot\partial_p B^{sm})\dot\partial_j\dot\partial_{h]}\dot\partial_s B^{pr}\}+$$

$$\frac{\delta^i_{[k}}{n^2-1}\{\dot\partial_{h]}(\dot\partial_j B^{sm})\dot\partial_p\dot\partial_s B^{pr}-\dot\partial_{h]}(\dot\partial_p B^{sm})\dot\partial_j\dot\partial_s B^{pr}+(\dot\partial_j B^{sm})\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}$$

$$-(\dot\partial_p B^{sm})\dot\partial_{[h}\dot\partial_j\dot\partial_s B^{pr}\}+\frac{x^i\dot\delta^i_{[k}}{n^2-1}\Big[\dot\partial_j(\dot\partial_{h]}\dot\partial_l B^{sm})\dot\partial_p\dot\partial_s B^{pr}-\dot\partial_j\dot\partial_{h]}(\dot\partial_p B^{sm})\dot\partial_l\dot\partial_s B^{pr}$$

$$+\dot{\partial}_{h|}\left(\dot{\partial}_l B^{sm}\right)\dot{\partial}_j\dot{\partial}_p\dot{\partial}_s B^{pr}-\dot{\partial}_{h|}\left(\dot{\partial}_p B^{sm}\right)\dot{\partial}_j\dot{\partial}_l\dot{\partial}_s B^{pr}+\dot{\partial}_j\left(\dot{\partial}_l B^{sm}\right)\dot{\partial}_{h|}\left(\dot{\partial}_p\dot{\partial}_s B^{pr}\right)$$

$$-\dot{\partial}_j\left(\dot{\partial}_p B^{sm}\right)\dot{\partial}_{h|}\left(\dot{\partial}_l\dot{\partial}_s B^{pr}\right)+\left(\dot{\partial}_l B^{sm}\right)\dot{\partial}_j\dot{\partial}_{h|}\dot{\partial}_p\dot{\partial}_s B^{pr}-\left(\dot{\partial}_p B^{sm}\right)\dot{\partial}_j\dot{\partial}_{h|}\dot{\partial}_l\dot{\partial}_s B^{pr}\Big\}\Big]$$

$$+\frac{2e^{2\sigma}g_{iu}\delta^i_{[-k}}{n^2-1}\Big[\Big\{\sigma_{m(j)}\dot{\partial}_{h]}\left(\dot{\partial}_p B^{pm}\right)-\sigma_{m(p)}\dot{\partial}_{h]}\left(\dot{\partial}_j B^{pm}\right)\Big\}$$

$$+\Big\{\dot{x}^l\sigma_{m(l)}\dot{\partial}_j\left(\dot{\partial}_{h|}\dot{\partial}_p B^{pm}\right)-\dot{x}^l\sigma_{m(p)}\dot{\partial}_j\dot{\partial}_{h|}\left(\dot{\partial}_l B^{pm}\right)\Big\}\Big] \tag{14}$$

where

$$\overline{W}_{jukh}=\overline{g}_{iu}\overline{W}^i_{jkh}\,. \tag{15}$$

We have the following indentities.

Theorem 1. When F_n *and* $\overline{F}_n$ are in conformal correspondence, we have

$$2\overline{W}^i_{[j<k>h}=2W^i_{[j<k>h]}+2\sigma_m\Big[\dot{\partial}_{[j}\left(\dot{\partial}^{jh]}_{h]}B^{im}\right)_{(k)l}-\dot{\partial}_{[j}\left(\dot{\partial}_k B^{im}\right)_{(h)]}+\frac{\dot{x}^l}{n+1}\Big\{\dot{\partial}_{[j}\dot{\partial}_k\left(\dot{\partial}_p B^{pm}\right)_{(h)]}$$

$$-\dot{\partial}_{[j}\dot{\partial}_{h]}\left(\dot{\partial}_p B^{pm}\right)_{(k)}\Big\}+\frac{\delta^i_{[j}}{n^2-1}\Big\{\dot{\partial}_p\left(\dot{\partial}_k B^{pm}\right)_{(h)]}-\dot{\partial}_p\left(\dot{\partial}_{h]}B^{pm}\right)_{(k)]}-\left(\dot{\partial}_k B^{pr}\right)G^m_{h]pr}$$

$$+\left(\dot{\partial}_{h]}B^{pr}\right)G^m_{kpr}\Big\}-\frac{\delta^i_k}{n^2-1}\Big\{n\dot{\partial}_{[j}\left(\dot{\partial}_{h]}B^{pm}\right)_{(p)}-n\dot{\partial}_{[j}\left(\dot{\partial}_p B^{pm}\right)_{(h])}+\dot{\partial}_{[h}\left(\dot{\partial}_{j]}B^{pm}\right)_{(p)}$$

$$-\dot{\partial}_{[h}\left(\dot{\partial}_p B^{pm}\right)_{(j])}\Big\}+\dot{x}^l\dot{\partial}_{[j}\dot{\partial}_{h]}\left(\dot{\partial}_l B^{pm}\right)_{(p)}-\dot{x}^l\dot{\partial}_{[j}\dot{\partial}_{h]}\left(\dot{\partial}_p B^{pm}\right)_{(l)}\Big\}+\frac{\delta^i_{[h}}{n^2-1}\Big\{n\dot{\partial}_{j]}\left(\dot{\partial}_K B^{pm}\right)_{(p)}$$

$$-n\dot{\partial}_{j]}\left(\dot{\partial}_p B^{pm}\right)_{(k)}+\dot{\partial}_k\left(\dot{\partial}_{j]}B^{pm}\right)_{(p)}+\dot{\partial}_k\left(\dot{\partial}_p B^{pm}\right)_{(j)]}+\dot{x}^l\dot{\partial}_{j]}\dot{\partial}_k\left(\dot{\partial}_l B^{pm}\right)_{(p)}$$

$$-\dot{x}^l\dot{\partial}_{j]}\dot{\partial}_k\left(\dot{\partial}_p B^{pm}\right)_{(l)}\Big\}+2\sigma_{m(k)}\Big\{\dot{\partial}_{[h}\left(\dot{\partial}_{j]}B^{im}\right)-\frac{\dot{x}^i}{n+1}\dot{\partial}_{[j}\left(\dot{\partial}_p B^{pm}\right)-\frac{\delta^i_{[j}}{n^2-1}\dot{\partial}_{h]}\left(\dot{\partial}_p B^{pm}\right)$$

$$-n\frac{\delta^i_{[h}}{n^2-1}\dot{\partial}_{j]}\left(\dot{\partial}_p B^{pm}\right)\Big\}-2\sigma_{m[h}\Big\{\dot{\partial}_k\left(\dot{\partial}_{j]}B^{im}\right)-\frac{\dot{x}^i}{n+1}\dot{\partial}_{j]}\left(\dot{\partial}_k\left(\dot{\partial}_p B^{pm}\right)-\frac{\delta^i_{j]}}{n^2-1}\dot{\partial}_k\left(\dot{\partial}_p B^{pm}\right)\right.$$

$$-2\sigma_{m[h}\Big\{\dot{\partial}_k\left(\dot{\partial}_{j]}B^{im}\right)-\frac{\dot{x}^i}{n+1}\dot{\partial}_{j]}\Big(\dot{\partial}_k\left(\dot{\partial}_p B^{pm}\right)-\frac{\delta^i_{j]}}{n^2-1}\dot{\partial}_k\left(\dot{\partial}_p B^{pm}\right)-n\frac{\delta^i_{[h}}{n^2-1}\dot{\partial}_{j]}\left(\dot{\partial}_p B^{pm}\right)\Big)$$

$$+2n\sigma_{m(p)}\Big\{\frac{\delta^i_{[h}}{n^2-1}\dot{\partial}_{k]}\left(\dot{\partial}_k B^{pm}\right)-\frac{\delta^i_{[k}}{n^2-1}\dot{\partial}_{[j}\left(\dot{\partial}_{h]}B^{pm}\right)\Big\}2\sigma_m\sigma_r\Big[\dot{\partial}_{[j}\left(\dot{\partial}_k B^{sm}\right)\dot{\partial}_{h]}\dot{\partial}_s B^{ir}-$$

$$-\frac{\dot{x}^i}{n+1}\Big\{\dot{\partial}_{[j}\dot{\partial}_k\left(\dot{\partial}_p B^{sm}\dot{\partial}_{h]}\dot{\partial}_s B^{pr}-\dot{\partial}_{[h}\left(\dot{\partial}_p B^{sm}\right)\dot{\partial}_{j]}\dot{\partial}_k\dot{\partial}_s B^{pr}+\dot{\partial}_{[j}\left(\dot{\partial}_k B^{sm}\right)\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_s B^{pr}\right.$$

$$+\left(\dot{\partial}_{[j}B^{sm}\right)\dot{\partial}_{h]}\dot{\partial}_k\left(\dot{\partial}_p B^{pr}\right)\Big\}+\frac{\delta^i_{[j}}{n^2-1}\Big\{\dot{\partial}_p\left(\dot{\partial}_{h]}B^{sm}\right)\dot{\partial}_k\dot{\partial}_s B^{pr}-\dot{\partial}_p\left(\dot{\partial}_k B^{sm}\right)\dot{\partial}_{h]}\dot{\partial}_s B^{pr}\Big\}$$

$$-\frac{\delta^l_{[h}}{n^2-1}\Big\{n\dot{\partial}_{j]}\dot{\partial}_k B^{sm}\Big)\dot{\partial}_p\dot{\partial}_s B^{pr} - n\dot{\partial}_{j]}\Big(\dot{\partial}_p B^{sm}\Big)\dot{\partial}_p\dot{\partial}_s B^{pr} + n\Big(\dot{\partial}_k B^{sm}\Big)\dot{\partial}_{j]}\dot{\partial}_p\dot{\partial}_s B^{pr}$$

$$-n\Big(\dot{\partial}_{j]}\dot{\partial}_k B^{sm}\Big)\dot{\partial}_p\dot{\partial}_s B^{pr} + \dot{\partial}_k\Big(\dot{\partial}_{j]} B^{sm}\Big)\dot{\partial}_p\dot{\partial}_s B^{pr} - \dot{\partial}_k\Big(\dot{\partial}_p B^{sm}\Big)\dot{\partial}_{j]}\dot{\partial}_s B^{pr}$$

$$+\dot{\partial}_{j]} B^{sm}\dot{\partial}_k\dot{\partial}_p\dot{\partial}_s B^{pr} - \Big(\dot{\partial}_p B^{sm}\Big)\dot{\partial}_k\dot{\partial}_{j]}\dot{\partial}_s B^{pr} - \dot{x}^l\dot{\partial}_{j]}\dot{\partial}_k\Big(\dot{\partial}_l B^{sm}\Big)\Big(\dot{\partial}_p\dot{\partial}_s B^{pr}\Big)$$

$$+\dot{x}^l\dot{\partial}_{j]}\dot{\partial}_k\Big(\dot{\partial}_p B^{sm}\Big)\Big(\dot{\partial}_l\dot{\partial}_s B^{pr}\Big) + \dot{x}^l\dot{\partial}_k\Big(\dot{\partial}_l B^{sm}\Big)\dot{\partial}_{j]}\Big(\dot{\partial}_p\dot{\partial}_s B^{pr}\Big) - \dot{x}^l\dot{\partial}_k\Big(\dot{\partial}_p B^{sm}\Big)\dot{\partial}_{j]}\Big(\dot{\partial}_l\dot{\partial}_s B^{pr}\Big)$$

$$+\dot{x}^l\dot{\partial}_{j]}\Big(\dot{\partial}_l B^{sm}\Big)\dot{\partial}_k\dot{\partial}_p\dot{\partial}_s B^{pr}\Big) - \dot{x}^l\dot{\partial}_{j]}\Big(\dot{\partial}_p B^{sm}\Big)\dot{\partial}_k\dot{\partial}_l\dot{\partial}_s B^{pr}\Big) + \dot{x}^l\Big(\dot{\partial}_l B^{sm}\Big)\dot{\partial}_{j]}\dot{\partial}_k\dot{\partial}_p\dot{\partial}_s B^{pr}$$

$$-\dot{x}\Big(\dot{\partial}_p B^{sm}\Big)\dot{\partial}_{j]}\dot{\partial}_k\dot{\partial}_l\dot{\partial}_s B^{pr}\Big\} - \frac{\delta^i_k}{n^2-1}\Big\{n\dot{\partial}_{[j}\Big(\dot{\partial}_p B^{sm}\Big)\Big(\dot{\partial}_{h]}\dot{\partial}_s B^{pr}\Big) - n\Big(\dot{\partial}_{[h} B^{sm}\Big)\dot{\partial}_{j]}\dot{\partial}_p\dot{\partial}_s B^{pr}$$

$$-\dot{\partial}_{[h}\Big(\dot{\partial}_p B^{sm}\Big)\dot{\partial}_{j]}\dot{\partial}_s B^{pr} - \Big(\dot{\partial}_{[j} B^{sm}\Big)\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_s B^{pr} - \dot{x}^l\dot{\partial}_{[h}\Big(\dot{\partial}_l B^{sm}\Big)\dot{\partial}_{j]}\Big(\dot{\partial}_p\dot{\partial}_s B^{pr}\Big)+$$

$$\dot{x}^l\dot{\partial}_{[h}\Big(\dot{\partial}_p B^{sm}\Big)\dot{\partial}_{j]}\dot{\partial}_l\dot{\partial}_s B^{pr} - \dot{x}^l\dot{\partial}_{[j}\dot{\partial}_{[h}\Big(\dot{\partial}_l B^{sm}\Big)\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_s B^{pr} + \dot{x}^l\dot{\partial}_{[j}\Big(\dot{\partial}_p B^{sm}\Big)\dot{\partial}_{h]}\Big(\dot{\partial}_l\dot{\partial}_s B^{pr}\Big)\Big\}\Big]$$

$$+2\sigma_{m l(j)}\left\{\frac{\delta^i_k}{n^2-1}\dot{\partial}_{h]}\Big(\dot{\partial}_p B^{pm}\Big) - \frac{\delta^l_{[h}}{n^2-1}\dot{\partial}_k\Big(\dot{\partial}_p B^{pm}\Big)\right\} + 2\frac{\delta^i_{[h}}{n^2-1}\sigma_{m(p)}\dot{\partial}_k\Big(\dot{\partial}_{j]} B^{pm}\Big)$$

$$-2\dot{x}^l\frac{\delta^i_{[h}}{n^2-1}\Big\{\sigma_{m(l)}\dot{\partial}_{j]}\Big(\dot{\partial}_k\dot{\partial}_p B^{pm}\Big) - \sigma_{m(p)}\dot{\partial}_{j]}\Big(\dot{\partial}_k\dot{\partial}_l B^{pm}\Big)\Big\} \qquad ...(16)$$

Proof. Interchange the indices k and h in (13) and subtracting the equation thus obtained from (13) and using the symmetric property of the function G^i_{jkh}, we get the result (16).

Theorem 2. When F_n and $\overline{F}_n$ are in conformal correspondence, we have

$$\overline{W}_{jukh} - \overline{W}_{jhku} = e^{2\sigma}\Big(W_{jukh} - W_{jkhu}\Big) + 2e^{2\sigma}\sigma_m g_{i[u]}\Big[\Big\{\Big(\dot{\partial}_k B^{ir}\Big)G^m_{h]jr} - \dot{\partial}_{h]} B^{ir}\Big)G^m_{kjr}$$

$$-\dot{\partial}_j\Big(\dot{\partial}_k B^{im}\Big)_{(h)} + \dot{\partial}_j\Big(\dot{\partial}_{h]} B^{im}\Big)_{(k)} + \frac{\dot{x}^i}{n+1}\Big\{\dot{\partial}_j\dot{\partial}_k\Big(\dot{\partial}_p B^{pm}\Big)_{h]}\Big\} - \dot{\partial}_j\dot{\partial}_{h]}\Big(\dot{\partial}_p B^{pm}\Big)_k\Big\}$$

$$+\frac{\delta^i_j}{n^2-1}\Big\{\dot{\partial}_p\Big(\dot{\partial}_k B^{pm}\Big)_{(h)]} - \dot{\partial}_p\Big(\dot{\partial}_{h]} B^{pm}\Big)_{(k)} - \dot{\partial}_k B^{pr}\Big)G^m_{h]pr} + \dot{\partial}_{h]} B^{pr}\Big)G^m_{kpr}\Big\}$$

$$+\frac{\delta^i_k}{n^2-1}\Big\{n\dot{\partial}_j\Big(\dot{\partial}_p B^{pm}\Big)_{(h)]} - n\Big(\dot{\partial}_j\dot{\partial}_{h]} B^{pm}\Big)_{(p)} - \dot{\partial}_{h]}\Big(\dot{\partial}_j B^{pm}\Big)_{(p)]} + \dot{\partial}_{h]}\Big(\dot{\partial}_p B^{pm}\Big)_{(j)}$$

$$-\dot{x}^l\dot{\partial}_j\Big(\dot{\partial}_{h]}\Big(\dot{\partial}_l B^{pm}\Big)_{(p)} + \dot{x}^l\dot{\partial}_j\dot{\partial}_{h]}\Big(\dot{\partial}_p B^{pm}\Big)_{(l)}\Big\} + \frac{\delta^i_{h]}}{n^2-1}\Big\{n\dot{\partial}_j\Big(\dot{\partial}_k B^{pm}\Big)_{(p)}$$

$$-n\dot{\partial}_j\Big(\dot{\partial}_p B^{pm}\Big)_{(k)} + \dot{\partial}_k\Big(\dot{\partial}_j B^{pm}\Big)_{(p)]} - \dot{\partial}_k\Big(\dot{\partial}_p B^{pm}\Big)_{(j)} + \dot{x}^l\dot{\partial}_j\Big(\dot{\partial}_k\Big(\dot{\partial}_l B^{pm}\Big)_{(p)}\Big)$$

$$+\dot{x}^l\dot{\partial}_j\dot{\partial}_k\Big(\dot{\partial}_p B^{pm}\Big)_{(l)}\Big\} + 2e^{2\sigma}g_{i[u}\Big[\sigma_{m(k)}\Big\{\dot{\partial}_{h]}\Big(\dot{\partial}_j B^{im}\Big) - \frac{\dot{x}}{n+1}\dot{\partial}_j\dot{\partial}_{h]}\Big(\dot{\partial}_p B^{pm}\Big)$$

$$-\frac{\delta^i_j}{n^2-1}\dot\partial_{h]}\left(\dot\partial_p B^{pm}\right)-n\frac{\delta^i_{h]}}{n^2-1}\dot\partial_j\left(\dot\partial_p B^{pm}\right)\Big\}-\sigma_{m(h)]}\Big\{\dot\partial_k\left(\dot\partial_j B^{im}\right)$$

$$-\frac{\dot x^i}{n+1}\dot\partial_j\dot\partial_k\left(\dot\partial_p B^{pm}\right)-\frac{\delta^i_j}{n^2-1}\dot\partial_k\left(\dot\partial_p B^{pm}\right)-n\frac{\delta^i_k}{n^2-1}\dot\partial_j\left(\dot\partial_p B^{pm}\right)\Big\}-n\frac{\delta^i_k}{n^2-1}$$

$$\sigma_{m(p)}\Big\{\dot\partial_j\left(\dot\partial_{h]}B^{pm}\right)\Big\}+\Big\{\dot\partial_j\left(\dot\partial_k B^{pm}\right)\Big\}\frac{n\delta^i_{h]}}{n^2-1}\Bigg]+2e^{2\sigma}g_{i[\mu}\sigma_m\sigma_r\Big[\dot\partial_j\left(\dot\partial_k B^{sm}\right)\dot\partial_{h]}\dot\partial_s B^{ir}$$

$$-\dot\partial_j\left(\dot\partial_{h]}B^{sm}\right)\dot\partial_k\dot\partial_s B^{ir}-\frac{\dot x^i}{n+1}\Big\{\dot\partial_j\dot\partial_k\left(\dot\partial_p B^{sm}\right)\dot\partial_{h]}\dot\partial_s B^{pr}-\dot\partial_j\dot\partial_{h]}\left(\dot\partial_p B^{sm}\right)\dot\partial_k\dot\partial_s B^{pr}$$

$$+\dot\partial_k\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_{h]}\left(\dot\partial_s B^{pr}\right)-\dot\partial_{h]}\left(\dot\partial_p B^{sm}\right)\dot\partial_j\left(\dot\partial_k\dot\partial_s B^{pr}\right)+\dot\partial_j\left(\dot\partial_k B^{sm}\right)\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}$$

$$-\dot\partial_j\left(\dot\partial_{h]}B^{sm}\right)\dot\partial_k\dot\partial_p\dot\partial_s B^{pr}+\left(\dot\partial_k B^{sm}\right)\dot\partial_j\dot\partial_{h]}\dot\partial_p\left(\dot\partial_s B^{pr}\right)-\left(\dot\partial_{h]}B^{sm}\right)\dot\partial_j\dot\partial_k\dot\partial_p\left(\dot\partial_s B^{pr}\right)$$

$$+\frac{\delta_j}{n+1}\Big\{\dot\partial_p\left(\dot\partial_{h]}B^{sm}\right)\dot\partial_k\dot\partial_s B^{pr}-\dot\partial_p\left(\dot\partial_k B^{sm}\right)\dot\partial_{h]}\dot\partial_s B^{pr}\Big\}+\frac{\delta^i_k}{n^2-1}\Big\{n\dot\partial_j\left(\dot\partial_{h]}B^{sm}\right)\dot\partial_p\dot\partial_s B^{pr}$$

$$-n\dot\partial_j\left(\dot\partial_p B^{sm}\right)\dot\partial_{h]}\dot\partial_s B^{pr}+n\dot\partial_{h]}\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_s B^{pr}-n\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_{h]}\dot\partial_s B^{pr}$$

$$+\dot\partial_{h]}\left(\dot\partial_j B^{sm}\right)\dot\partial_p\dot\partial_s B^{pr}-\dot\partial_{h]}\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_s B^{pr}+\left(\dot\partial_j B^{sm}\right)\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}$$

$$-\left(\dot\partial_p B^{sm}\right)\dot\partial_{h]}\dot\partial_j\dot\partial_s B^{pr}+\dot x^l\dot\partial_j\dot\partial_{h]}\left(\dot\partial_l B^{sm}\right)\dot\partial_p\dot\partial_s B^{pr}-\dot x^l\dot\partial_j\dot\partial_{h]}\left(\dot\partial_p B^{sm}\right)\dot\partial_l\dot\partial_s B^{pr}$$

$$+\dot x^l\dot\partial_{h]}\left(\left(\dot\partial_l B^{sm}\right)\dot\partial_j\dot\partial_p\dot\partial_s B^{pr}-\dot x^l\dot\partial_{h]}\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_l\dot\partial_s B^{pr}+\dot x^l\dot\partial_j\left(\dot\partial_l B^{sm}\right)\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}\right.$$

$$-\dot x^l\dot\partial_j\left(\dot\partial_p B^{sm}\right)\dot\partial_{h]}\dot\partial_l\dot\partial_s B^{pr}+\dot x^l\left(\dot\partial_l B^{sm}\right)\dot\partial_j\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}-\dot x^l\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_{h]}\dot\partial_l\dot\partial_s B^{pr}\Big\}$$

$$-\frac{\delta^i_{h]}}{n^2-1}\Big\{n\dot\partial_j\left(\dot\partial_k B^{sm}\right)\dot\partial_p\dot\partial_s B^{pr}-n\dot\partial_j\left(\dot\partial_p B^{sm}\right)\dot\partial_k\dot\partial_s B^{pr}+n\dot\partial_k\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_s B^{pr}$$

$$-n\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_k\dot\partial_s B^{pr}+\dot\partial_k\left(\dot\partial_j B^{sm}\right)\dot\partial_p\dot\partial_s B^{pr}-\dot\partial_k\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_s B^{pr}$$

$$+\left(\dot\partial_j B^{sm}\right)\dot\partial_k\dot\partial_p\dot\partial_s B^{pr}-\left(\dot\partial_p B^{sm}\right)\dot\partial_k\dot\partial_j\dot\partial_s B^{pr}+\dot x\dot\partial_j\dot\partial_k\left(\dot\partial_l B^{sm}\right)\dot\partial_p\dot\partial_s B^{pr}$$

$$-\dot x^l\dot\partial_j\dot\partial_k\left(\dot\partial_p B^{sm}\right)\dot\partial_l\dot\partial_s B^{pr}+\dot x^l\dot\partial_k\left(\left(\dot\partial_l B^{sm}\right)\dot\partial_j\dot\partial_p\dot\partial_s B^{pr}-\dot x^l\dot\partial_k\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_l\dot\partial_s B^{pr}\right.$$

$$+\dot x^l\dot\partial_j\left(\dot\partial_l B^{sm}\right)\dot\partial_k\dot\partial_p\dot\partial_s B^{pr}-\dot x^l\dot\partial_j\left(\dot\partial_p B^{sm}\right)\dot\partial_k\dot\partial_l\dot\partial_s B^{pr}+\dot x^l\left(\dot\partial_l B^{sm}\right)\dot\partial_j\dot\partial_k\dot\partial_p\dot\partial_s B^{pr}$$

$$-\dot x^l\left(\dot\partial_p B^{sm}\right)\dot\partial_j\dot\partial_k\dot\partial_l\dot\partial_s B^{pr}\Big\}\Big]+2e^{2\sigma}g_{i[u}\Big[\sigma_{m(j)}\Big\{\frac{\delta^i_k}{n^2-1}\dot\partial_{h]}\left(\dot\partial_p B^{pm}\right)-\frac{\delta^i_{h]}}{n^2-1}\dot\partial_k\left(\dot\partial_p B^{pm}\right)\Big\}$$

$$+\sigma_{m(p)}\Big\{\frac{\delta^i_k}{n^2-1}\dot\partial_{h]}\left(\dot\partial_j B^{pm}\right)-\frac{\delta^i_{h]}}{n^2-1}\dot\partial_k\left(\dot\partial_j B^{pm}\right)\Big\}+\dot x^l\sigma_{m(l)}\Big\{\frac{\delta^i_k}{n^2-1}\dot\partial_j\dot\partial_{h]}\left(\dot\partial_p B^{pm}\right)$$

$$-\frac{\delta^i_{h]}}{n^2-1}\dot\partial_j\dot\partial_k\left(\dot\partial_p B^{pm}\right)\Big\}-\dot x^l\sigma_{m(p)}\Big\{\frac{\delta^i_k}{n^2-1}\dot\partial_j\dot\partial_{h]}\left(\dot\partial_l B^{pm}\right)+\frac{\delta^i_{h]}}{n^2-1}\dot\partial_j\dot\partial_k\left(\dot\partial_l B^{pm}\right)\Big\}\Big] \quad (17)$$

Proof. Interchange the indices u and h in (14) and subtracting the equation thus obtained to (14), we obtain the identity (17).

Theorem 3. When F_n and $\overline{F}_n$ are in conformal correspondence, we have

$$\overline{W}_{jukh} + \overline{W}_{ujkh} = e^{2\sigma}\left(W_{jukh} + W_{ujhk}\right) + 4e^{2\sigma}\sigma_m \Big[g_{i(u)}\Big\{\big(\dot\partial_{[k}B^{ir}\big)G^m_{h]j)r} - \dot\partial_{j)}\big(\dot\partial_{[k}B^{im}\big)$$

$$+ \frac{g_{i(u}\dot x^l}{n+1}\dot\partial_{j)}\dot\partial_{[k}\big(\dot\partial_p B^{pm}\big)_{(h)]} + \frac{g_{i(u}\delta^i_{j)}}{n+1}\Big\{\dot\partial_p\big(\dot\partial_{[k}B^{pm}\big)_{(h)]} - \big(\dot\partial_{[k}B^{pr}\big)G^m_{h]pr}\Big\}$$

$$- \frac{g_{i(u}\delta^i_{[k}}{n^2-1}\Big\{n\dot\partial_{j)}\big(\dot\partial_{h]}B^{pm}\big)_{(p)} - n\dot\partial_{j)}\big(\dot\partial_p B^{pm}\big)_{(h)]} + \dot\partial_{h]}\big(\dot\partial_{j)}B^{pm}\big)_{(p)} - \dot\partial_{h]}\big(\dot\partial_p B^{pm}\big)_{(j))}$$

$$+ \dot x^l \dot\partial_{j)}\dot\partial_{h]}\big(\dot\partial_l B^{pm}\big)_{(p)} - \dot\partial_{j)}\dot\partial_{h]}\big(\dot\partial_p B^{pm}\big)_{(l)}\Big\}\Big] + 4e^{2\sigma}\sigma_{m|(k)}g_{i(u)}\Big\{\dot\partial_{h]}\big(\dot\partial_{j)}B^{im}\big)$$

$$- \frac{\dot x^l}{n+1}\dot\partial_{j)}\big(\dot\partial_{h]}\dot\partial_p B^{pm}\big) - \frac{\delta^i_{j)}}{n^2-1}\dot\partial_{h]}\big(\dot\partial_p B^{pm}\big)\Big\} + 4e^{2\sigma}n\frac{g_{i(u}\delta^i_{[k}}{n^2-1}\Big\{\sigma_{m(h)]}\dot\partial_{j)}\big(\dot\partial_p B^{pm}\big)$$

$$- \sigma_{m(p)}\big(\dot\partial_{j)}\dot\partial_{h]}B^{pm}\big)\Big\} + 4e^{2\sigma}\sigma_m\sigma_r g_{i(u}\Big[\Big\{\dot\partial_{j)}\big(\dot\partial_{[k}B^{sm}\big)\dot\partial_{h]}\dot\partial_s B^{ir} - \frac{\dot x^i}{n+1}\dot\partial_{j)}\dot\partial_{[k}$$

$$\big(\dot\partial_p B^{sm}\big)\dot\partial_{h]}\dot\partial_s B^{pr} + \dot\partial_{[k}\big(\dot\partial_p B^{sm}\big)\dot\partial_{j)}\big(\dot\partial_{h]}\dot\partial_s B^{pr}\big) + \dot\partial_{j)}\big(\dot\partial_{[k}B^{sm}\big)\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}$$

$$+ \big(\dot\partial_{[k}B^{sm}\big)\dot\partial_{j)}\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr} + \frac{\delta^i_{j)}}{n^2-1}\dot\partial_p\big(\dot\partial_{[h}B^{sm}\big)\dot\partial_{k]}\dot\partial_s B^{pr} + \frac{\delta^i_{[k}}{n^2-1}\Big\{n\dot\partial_{j)}\big(\dot\partial_{h]}B^{sm}\big)$$

$$\dot\partial_p\dot\partial_s B^{pr} - n\dot\partial_{j)}\big(\dot\partial_p B^{sm}\big)\dot\partial_{h]}\dot\partial_s B^{pr} + n\big(\dot\partial_{h]}B^{sm}\big)\dot\partial_{j)}\dot\partial_p\dot\partial_s B^{pr} - n\big(\dot\partial_p B^{sm}\big)\dot\partial_{j)}\dot\partial_{h]}\dot\partial_s B^{pr}\big)$$

$$+ \dot\partial_{h]}\big(\dot\partial_{j)}B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr} - \dot\partial_{h]}\big(\dot\partial_p B^{sm}\big)\dot\partial_{j)}\dot\partial_s B^{pr} + \big(\dot\partial_{j)}B^{sm}\big)\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}$$

$$- \big(\dot\partial_p B^{sm}\big)\dot\partial_{h]}\dot\partial_{j)}\dot\partial_s B^{pr} + \dot x^l\dot\partial_{j)}\dot\partial_{h]}\big(\dot\partial_l B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr} - \dot\partial_{j)}\dot\partial_{h]}\big(\dot\partial_p B^{sm}\big)\dot\partial_l\dot\partial_s B^{pr}$$

$$+ \dot\partial_{h]}\big(\dot\partial_l B^{sm}\big)\dot\partial_{j)}\dot\partial_p\dot\partial_s B^{pr} - \dot\partial_{h]}\big(\dot\partial_p B^{sm}\big)\dot\partial_{j)}\dot\partial_l\dot\partial_s B^{pr} + \dot\partial_{j)}\big(\dot\partial_l B^{sm}\big)\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}$$

$$- \dot\partial_{j)}\big(\dot\partial_p B^{sm}\big)\dot\partial_{h]}\dot\partial_l\dot\partial_s B^{pr} + \big(\dot\partial_l B^{sm}\big)\dot\partial_{j)}\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr} - \big(\dot\partial_p B^{sm}\big)\dot\partial_{j)}\dot\partial_{h]}\dot\partial_l\dot\partial_s B^{pr}\big)\Big\}\Big]$$

$$+ 4e^{2\sigma}\frac{g_{(u}\delta_{[k}}{n-1}\Big[\Big\{\sigma_{m(j))}\dot\partial_{h]}\big(\dot\partial_p B^{pm}\big) - \sigma_{m(p)}\dot\partial_{h]}\big(\dot\partial_{j)}B^{pm}\big)\Big\}$$

$$+ \dot x^l\Big\{\sigma_{m(l)}\dot\partial_{j)}\big(\dot\partial_{h]}\dot\partial_p B^{sm}\big) - \sigma_{m(p)}\dot\partial_{j)}\big(\dot\partial_{h]}\dot\partial_l B^{pm}\big)\Big\}\Big]. \qquad \ldots(18)$$

Proof. Interchanging the indices in each pair (j,u) and (k,h) in equation (14) and adding the equations thus obtained from (14), we obtain the identity (18).

Theorem 4. When F_n and $\overline{F}_n$ are in conformal correspondence, we have

$$\overline{W}_{jukh} + \overline{W}_{jhku} + \overline{W}_{khju} + \overline{W}_{kujh} = e^{2\sigma}\left(W_{jhku} + W_{jukh} + W_{kujh} + W_{khju}\right) + 4e^{2\sigma}\sigma_m g_{i(u}$$

$$[\dot\partial_{|k} B^{ir}) G^m_{h)j|r} - \dot\partial_{[j}\left(\dot\partial_{k]} B^{im}\right)_{(h))} + \dot\partial_{[j}\left(\dot\partial_{h)} B^{im}\right)_{(k)]} + \frac{\dot x^i}{n+1}\{\dot\partial_{[j}\dot\partial_{k]}\left(\dot\partial_p B^{pm}\right)_{(h))}$$

$$-\frac{\dot x^i}{n+1}\left\{\dot\partial_{[j}\dot\partial_{h)}\left(\dot\partial_p B^{pm}\right)_{(k)]}+\frac{\delta_{[j}}{n+1}\{\dot\partial_p\left(\dot\partial_{k]} B^{pm}\right)_{(h))} - \dot\partial_{[j}\left(\dot\partial_{k]} B^{im}\right)_{(h))}\right.$$

$$+\dot\partial_{[j}\left(\dot\partial_{h)} B^{im}\right)_{(k)]} + \frac{\dot x^i}{n+1}\left\{\dot\partial_{[j}\dot\partial_{k]}\left(\dot\partial_p B^{pm}\right)_{(h))} - \frac{\dot x^i}{n+1}\{\dot\partial_{[j}\dot\partial_{h)}\left(\dot\partial_p B^{pm}\right)_{(k)]}\right.$$

$$+\frac{\delta^i_{[j}}{n+1}\{\dot\partial_p\left(\dot\partial_{k]} B^{pm}\right)_{(h))} - \dot\partial_p\left(\dot\partial_{h)} B^{pm}\right)_{(k)]} - \left(\dot\partial_{k]} B^{pr}\right) G^m_{h)pr} + \left(\dot\partial_{h)} B^{pr}\right) G^m_{k|pr}\}$$

$$-\frac{\delta^i_{[k}}{n^2-1}\{n\dot\partial_{j]}\left(\dot\partial_{h)} B^{pm}\right)_{(p)} - n\dot\partial_{j]}\left(\dot\partial_p B^{pm}\right)_{(h))} + \dot\partial_{h)}\left(\dot\partial_{j]} B^{pm}\right)_{(p)}$$

$$-\dot\partial_{h)}\left(\dot\partial_p B^{pm}\right)_{(j)]}\Big\} + \frac{\delta^i_{h)}}{n^2-1}\{n\dot\partial_{[j}\left(\dot\partial_{k]} B^{pm}\right)_{(p)} - n\dot\partial_{[j}\left(\dot\partial_p B^{pm}\right)_{(k)]}$$

$$+\dot\partial_{[k}\left(\dot\partial_{j]} B^{pm}\right)_{(p)} - \dot\partial_{[k}\left(\dot\partial_p B^{pm}\right)_{(j)]}\Big\} - \frac{\dot x^l\delta^i_{[k}}{n^2-1}\dot\partial_{j]}\{\dot\partial_{h)}\left(\dot\partial_l B^{pm}\right)_{(p)}$$

$$-\dot\partial_{h)}\left(\dot\partial_p B^{pm}\right)_{(j)]}\Big\} + \frac{\dot x^l\delta^i_{h)}}{n^2-1}\dot\partial_{[j}\left(\dot\partial_l B^{pm}\right)_{(p)} - \dot\partial_{[k}\left(\dot\partial_p B^{pm}\right)_{(j)]}\Big]\Big]$$

$$+4e^{2\sigma} g_{i(u}\Big[\sigma_{m[k}\Big\{\dot\partial_{h)}\left(\dot\partial_{j]} B^{im}\right) - \frac{\dot x^l}{n+1}\dot\partial_{j]}\left(\dot\partial_{h)}\dot\partial_p B^{pm}\right) - \frac{\delta^i_{j]}}{n+1}\dot\partial_{h)}\left(\dot\partial_p B^{pm}\right)\Big\}$$

$$-\sigma_{m(h))}\Big\{\dot\partial_{[k}\left(\dot\partial_{j]} B^{im}\right) + \frac{\dot x^i}{n+1}\dot\partial_{[j}\left(\dot\partial_{k]}\dot\partial_p B^{pm}\right) + \frac{\delta^i_{[j}}{n+1}\dot\partial_{k]}\left(\dot\partial_p B^{pm}\right)\Big\}$$

$$+\frac{n\delta^i_{[k}}{n^2-1}\{\sigma_{m(h))}\left(\dot\partial_{j]}\dot\partial_p B^{pm}\right) + \frac{n\delta^i_{[k}}{n^2-1}\{\sigma_{m(h))}\left(\dot\partial_{j]}\dot\partial_p B^{pm}\right)$$

$$-\sigma_{m(p)}\left(\dot\partial_{j]}\dot\partial_{h)} B^{pm}\right)\Big\} - \frac{n\delta^i_{h)}}{n^2-1}\{\sigma_{m[(k)}\left(\dot\partial_{j]}\dot\partial_p B^{pm}\right) - \sigma_{m(p)}\left(\dot\partial_{[j}\dot\partial_{k]} B^{pm}\right)\Big\}$$

$$+4e^{2\sigma}\sigma_m\sigma_r g_{i(u}\Big[\dot\partial_{[j}\left(\dot\partial_{k]} B^{sm}\right)\dot\partial_{h)}\dot\partial_s B^{ir} - \dot\partial_{[j}\left(\dot\partial_{h)} B^{sm}\right)\dot\partial_{k]}\dot\partial_s B^{ir}$$

$$-\frac{\dot x^i}{n+1}\{\dot\partial_{[j}\dot\partial_{k]}\left(\dot\partial_p B^{sm}\right)\dot\partial_{h)}\dot\partial_s B^{pr} - \dot\partial_{[j}\dot\partial_{h)}\left(\dot\partial_p B^{sm}\right)\dot\partial_{k]}\dot\partial_s B^{pr}$$

$$-\dot\partial_{[j}\dot\partial_{h)}\left(\dot\partial_p B^{sm}\right)\dot\partial_{k]}\dot\partial_s B^{pr} + \dot\partial_{[k}\left(\dot\partial_p B^{sm}\right)\dot\partial_{j]}\left(\dot\partial_{h)}\dot\partial_s B^{pr}\right) - \dot\partial_{h)}\left(\dot\partial_p B^{sm}\right)$$

$$\dot\partial_{[j}\left(\dot\partial_{k]}\dot\partial_s B^{pr}\right) + \dot\partial_{[j}\left(\dot\partial_{k]} B^{sm}\right)\dot\partial_{h)}\dot\partial_p\dot\partial_s B^{pr} - \dot\partial_{[j}\left(\dot\partial_{k]} B^{sm}\right)\dot\partial_{h)}\dot\partial_p\dot\partial_s B^{pr}$$

$$-\left(\dot\partial_{[k} B^{sm}\right)\dot\partial_{j]}\dot\partial_{h)}\left(\dot\partial_p\dot\partial_s B^{pr}\right) + \left(\dot\partial_{h)} B^{sm}\right)\dot\partial_{[j}\dot\partial_{k]}\left(\dot\partial_p\dot\partial_s B^{pr}\right)\Big\}$$

$$+\frac{\delta^i_{[j}}{n+1}\{\dot\partial_p\left(\dot\partial_{h)} B^{sm}\right)\dot\partial_{k]}\dot\partial_s B^{pr} - \dot\partial_p\left(\dot\partial_{k]} B^{sm}\right)\dot\partial_{h)}\dot\partial_s B^{pr}\}$$

$$+\frac{\delta^i_{[k}}{n^2-1}\Big\{n\big(\dot\partial_{j]}\dot\partial_{h)}B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr}-n\big(\dot\partial_{j]}\dot\partial_p B^{sm}\big)\dot\partial_{h)}\dot\partial_s B^{pr}$$

$$+n\big(\dot\partial_{h)}B^{sm}\big)\dot\partial_{j]}\dot\partial_p\dot\partial_s B^{pr}-n\big(\dot\partial_p B^{sm}\big)\dot\partial_{j]}\dot\partial_{h)}\dot\partial_s B^{pr}$$

$$+\dot\partial_{h)}\big(\dot\partial_{j]}B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr}-\dot\partial_{h)}\big(\dot\partial_p B^{sm}\big)\dot\partial_{j]}\dot\partial_s B^{pr}+\big(\dot\partial_{j]}B^{sm}\big)\dot\partial_{h)}\dot\partial_p\dot\partial_s B^{pr}$$

$$-\big(\dot\partial_p B^{sm}\big)\dot\partial_{h)}\dot\partial_{j]}\dot\partial_s B^{pr}\Big\}-\frac{\delta^i_{h)}}{n^2-1}\Big\{n\big(\dot\partial_{[j}\dot\partial_{k]}B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr}$$

$$-n\big(\dot\partial_{[j}\dot\partial_p B^{sm}\big)\dot\partial_{k]}\dot\partial_s B^{pr}+n\big(\dot\partial_{[k}B^{sm}\big)\dot\partial_{j]}\dot\partial_p\dot\partial_s B^{pr}-n\big(\dot\partial_p B^{sm}\big)$$

$$\dot\partial_{[j}\dot\partial_{k]}\dot\partial_s B^{pr}+\dot\partial_{[k}\big(\dot\partial_{j]}B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr}+\dot\partial_{[k}\big(\dot\partial_p B^{sm}\big)\dot\partial_{j]}\dot\partial_s B^{pr}+\big(\dot\partial_{[j}B^{sm}\big)\dot\partial_{k]}\dot\partial_p\dot\partial_s B^{pr}$$

$$-\big(\dot\partial_p B^{sm}\big)\dot\partial_{[k}\dot\partial_{j]}\dot\partial_s B^{pr}\Big\}+\frac{\dot x^l\delta^i_{[k}}{n^2-1}\Big\{\dot\partial_{j]}\big(\dot\partial_{h)}\dot\partial_l B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr}-\dot\partial_{[j}\dot\partial_{h)}\big(\dot\partial_p B^{sm}\big)\dot\partial_l\dot\partial_s B^{pr}$$

$$+\dot\partial_{h)}\big(\dot\partial_l B^{sm}\big)\dot\partial_{j]}\dot\partial_p\dot\partial_s B^{pr}-\dot\partial_{h)}\big(\dot\partial_p B^{sm}\big)\dot\partial_{j]}\dot\partial_l\dot\partial_s B^{pr}+\dot\partial_{j]}\big(\dot\partial_l B^{sm}\big)\dot\partial_{h)}\big(\dot\partial_p\dot\partial_s B^{pr}\big)$$

$$-\dot\partial_{j]}\big(\dot\partial_p B^{sm}\big)\dot\partial_{h)}\big(\dot\partial_l\dot\partial_s B^{pr}\big)+\big(\dot\partial_l B^{sm}\big)\dot\partial_{j]}\dot\partial_{h)}\dot\partial_p\dot\partial_s B^{pr}-\big(\dot\partial_p B^{sm}\big)\dot\partial_{j]}\dot\partial_{h)}\dot\partial_l\dot\partial_s B^{pr}\Big\}$$

$$-\frac{\dot x^l\delta^i_{h)}}{n^2-1}\Big\{\dot\partial_{[j}\big(\dot\partial_{k]}\dot\partial_l B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr}-\dot\partial_{[j}\dot\partial_{k]}\big(\dot\partial_p B^{sm}\big)\dot\partial_l\dot\partial_s B^{pr}+\dot\partial_{[k}\big(\dot\partial_l B^{sm}\big)\dot\partial_{j]}\dot\partial_p\dot\partial_s B^{pr}$$

$$-\dot\partial_{[k}\big(\dot\partial_p B^{sm}\big)\dot\partial_{j]}\dot\partial_l\dot\partial_s B^{pr}+\dot\partial_{ij}\big(\dot\partial_l B^{sm}\big)\dot\partial_{k]}\big(\dot\partial_p\dot\partial_s B^{pr}\big)-\dot\partial_{[j}\big(\dot\partial_p B^{sm}\big)\dot\partial_{k]}\big(\dot\partial_l\dot\partial_s B^{pr}\big)$$

$$+\big(\dot\partial_l B^{sm}\big)\dot\partial_{[j}\dot\partial_{k]}\dot\partial_p\dot\partial_s B^{pr}-\big(\dot\partial_p B^{sm}\big)\dot\partial_{[j}\dot\partial_{k]}\dot\partial_l\dot\partial_s B^{pr}\Big\}\Big]+4e^{2\sigma}g_{i(u}\Bigg[\frac{\delta^i_{[k}}{n^2-1}\Big\{\sigma_{m(j)]}\dot\partial_{h)}\big(\dot\partial_p B^{pm}\big)$$

$$-\sigma_{m(p)}\dot\partial_{h)}\big(\dot\partial_{j]}B^{pm}\big)+\dot x^l\sigma_{m(l)}\dot\partial_{j]}\big(\dot\partial_{h)}\dot\partial_p B^{pm}\big)-\dot x^l\sigma_{m(p)}\dot\partial_{j]}\dot\partial_{h)}\big(\dot\partial_l B^{pm}\big)\Big\}$$

$$-\frac{\delta^i_{h)}}{n^2-1}\Big\{\sigma_{m(j)}\dot\partial_{k]}\big(\dot\partial_p B^{pm}\big)-\sigma_{m(p)}\dot\partial_{[k}\big(\dot\partial_{j]}B^{pm}\big)+\dot x^l\sigma_{m(l)}\dot\partial_{[j}\big(\dot\partial_{k]}\dot\partial_p B^{pm}\big)$$

$$-\dot x^l\sigma_{m(p)}\dot\partial_{[j}\dot\partial_{k]}\big(\dot\partial_l B^{pm}\big)\Big\}-\frac{\dot x^l\delta^i_{[k}}{n^2-1}\dot\partial_j\Big\{\dot\partial_{h]}\big(\dot\partial_l B^{pm}\big)_{(p)}-\dot\partial_{h]}\big(\dot\partial_p B^{pm}\big)_{(l)}\Big\}\Bigg]$$

$$+2e^{2\sigma}g_{i:}\sigma_{m[k}\left\{\dot\partial_{h]}\big(\dot\partial_j B^{im}\big)-\frac{\dot x^i}{n+1}\dot\partial_j\big(\dot\partial_{h]}\dot\partial_p B^{pm}\big)-\frac{\delta^i_j}{n+1}\dot\partial_{h]}\big(\dot\partial_p B^{pm}\big)\right\}$$

$$+\frac{2n\delta^i_{[k}}{n^2-1}e^{2\sigma}g_{iu}\Big\{\sigma_{m(h)]}\big(\dot\partial_j\dot\partial_p B^{pm}\big)-\sigma_{m(p)}\big(\dot\partial_j\dot\partial_{h]}B^{pm}\big)\Big\}+2e^{2\sigma}g_{iu}\sigma_m\sigma_r\Big[\dot\partial_j\big(\dot\partial_{[k}B^{sm}\big)$$

$$\dot\partial_{h]}\dot\partial_s B^{ir}-\frac{\dot x^i}{n+1}\Big\{\dot\partial_j\dot\partial_{[k}\big(\dot\partial_p B^{sm}\big)\dot\partial_{h]}\dot\partial_s B^{pr}+\dot\partial_{[k}\big(\dot\partial_p B^{sm}\big)\dot\partial_j\big(\dot\partial_{h]}\dot\partial_s B^{pr}\big)$$

$$+\dot\partial_j\big(\dot\partial_{[k}B^{sm}\big)\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr}-\big(\dot\partial_{[k}B^{sm}\big)\dot\partial_j\dot\partial_{h]}\big(\dot\partial_p\dot\partial_s B^{pr}\big)\Big\}-\frac{\delta^i_j}{n+1}\Big\{\dot\partial_p\big(\dot\partial_{[h}B^{sm}\big)\dot\partial_{k]}\dot\partial_s B^{pr}\Big\}$$

$$+\frac{\delta^i_{[k}}{n^2-1}\Big\{n\big(\dot\partial_j\dot\partial_{h]}B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr}-n\big(\dot\partial_j\dot\partial_p B^{sm}\big)\dot\partial_{h]}\dot\partial_s B^{pr}+n\big(\dot\partial_{h]}B^{sm}\big)\dot\partial_j\dot\partial_p\dot\partial_s B^{pr}$$

$$
\begin{aligned}
&-n\left(\dot\partial_p B^{sm}\right)\dot\partial_j \partial_{h]}\dot\partial_s B^{pr} + \Big\{\dot\partial_{h]}\left(\dot\partial_j B^{sm}\right)\dot\partial_p \partial_s B^{pr} - \dot\partial_{h]}\left(\dot\partial_p B^{sm}\right)\dot\partial_j \dot\partial_s B^{pr} + \\
&+\left(\dot\partial_j B^{sm}\right)\dot\partial_{h]}\dot\partial_p \dot\partial_s B^{pr} - \left(\dot\partial_p B^{sm}\right)\dot\partial_{[h}\dot\partial_j \dot\partial_s B^{pr}\Big\} + \frac{\dot x^l \delta^i_{[k}}{n^2-1}\Big\{\dot\partial_j\left(\dot\partial_{h]}\dot\partial_l B^{sm}\right)\dot\partial_p \dot\partial_s B^{pr} \\
&-\dot\partial_j \dot\partial_{h]}\left(\dot\partial_p B^{sm}\right)\dot\partial_l \dot\partial_s B^{pr} + \dot\partial_{h]}\left(\dot\partial_l B^{sm}\right)\dot\partial_j \dot\partial_p \dot\partial_s B^{pr} - \dot\partial_{h]}\left(\dot\partial_p B^{sm}\right)\dot\partial_j \dot\partial_l \dot\partial_s B^{pr} \\
&+\dot\partial_j\left(\dot\partial_l B^{sm}\right)\dot\partial_{h]}\left(\dot\partial_p \dot\partial_s B^{pr}\right) - \dot\partial_j\left(\dot\partial_p B^{sm}\right)\dot\partial_{h]}\left(\dot\partial_l \dot\partial_s B^{pr}\right) + \left(\dot\partial_l B^{sm}\right)\dot\partial_j \dot\partial_{h]}\dot\partial_p \dot\partial_s B^{pr} \\
&-\left(\dot\partial_p B^{sm}\right)\dot\partial_j \dot\partial_{h]}\dot\partial_l \dot\partial_s B^{pr}\Big\}\Big] + \frac{2e^{2\sigma} g_{iu}\delta^i_{[k}}{n^2-1}\Big[\Big\{\sigma_{m(j)}\dot\partial_{h]}\left(\dot\partial_p B^{pm}\right) - \sigma_{m(p)}\dot\partial_{h]}\left(\dot\partial_j B^{pm}\right)\Big\} \\
&+\dot x^l\Big\{\sigma_{m(l)}\dot\partial_j\left(\dot\partial_{h]}\dot\partial_p B^{pm}\right) - \sigma_{m(p)}\dot\partial_j \dot\partial_{h]}\left(\dot\partial_l B^{pm}\right)\Big\}\Big]
\end{aligned}
\quad ...(19)
$$

Proof. The proof follows in consequence of (14) anad (17).

3. Decomposition of Conformal Projective Curvature Tensor. We considered the decomposition of conformal projective curvature tensor in the form

$$\overline{W}^i_{jkh} = \overline{X}^i \overline{\phi}_{jkh} \tag{20}$$

where $\overline{\phi}_{jkh}$ is a homogeneous conformal decomposition tensor and $\overline{X}^i$ is a non zoro conformal vector such

$$\overline{X}^i \overline{V}_i = 1 \tag{21}$$

Similar manner the decomosition of projective curvature tensor W^i_{jkh} in the form

$$W^i_{jkh} = X^i \phi_{jkh} \tag{22}$$

where the decomposition vector X^i also satisfies the condition

$$X^i V_i = 1 \tag{23}$$

Transvecting (22) by $\dot x^i$, we get

$$W^i_{kh} = X^i \phi_{kh} \tag{24}$$

where

$$\phi_{kh} = -\phi_{jhk}\dot x^j \tag{25}$$

The decomposition tensor ϕ_{jkh} satisfies the identity

$$\phi_{kh} = -\phi_{hk} \tag{26}$$

We notice that the decomposition vector $\overline{X}^i$ and the recurrence vector V_i

are transformed conformally as under :

$$\bar{X}^i = e^{-\sigma}X^i \tag{27}$$

and

$$\bar{V}_i = e^{\sigma}V_i \tag{28}$$

respectively.

Using equation (20) and (22) in equation (13), the obtained equation transvercted by $\bar{V}_i$ and using the equation (21), (23), (27) and (28), we get

$$\bar{\phi}_{jkh} = e^{\sigma}\phi_{jkh} + V_i 2\sigma_m\Big[\dot\partial_{[k}B^{ir}\Big)G^m_{h]jr} - \dot\partial_j\big(\dot\partial_{[k}B^{im}\big)_{(h)} + \frac{\dot x^i}{n+1}\Big\{\dot\partial_j\dot\partial_{[k}\big(\dot\partial_p B^{pm}\big)_{(h)]}$$

$$-\dot\partial_j\dot\partial_{[k}\big(\dot\partial_{h]}B^{pm}\big)_{(p)}\Big\} + \frac{\delta^i_j}{n+1}\Big\{\dot\partial_p\big(\dot\partial_{[k}B^{pm}\big)_{(h)]} - \big(\dot\partial_{[k}B^{pr}\Big)G^m_{h]pr}\Big\} - \frac{\delta^i_{[k}}{n^2-1}\Big\{n\dot\partial_j\big(\dot\partial_{h]}B^{pm}\big)_{(p)}$$

$$-n\dot\partial_j\big(\dot\partial_p B^{pm}\big)_{(h)]} + \dot\partial_{h]}\big(\dot\partial_j B^{pm}\big)_{(p)]} - \dot\partial_{h]}\big(\dot\partial_p B^{pm}\big)_{(j)]}\Big\} - \frac{\dot x^l\delta^i_{[k}}{n^2-1}\dot\partial_j\Big\{\dot\partial_{h]}\big(\dot\partial_l B^{pm}\big)_{(p)}$$

$$-\dot\partial_{h]}\big(\dot\partial_p B^{pm}\big)_{(l)}\Big\}\Big] + V_i 2\sigma_{m[k}\Big\{\dot\partial_{h]}\big(\dot\partial_j B^{im}\big) - \frac{\dot x^i}{n+1}\dot\partial_j\big(\dot\partial_{h]}\dot\partial_p B^{pm}\big)$$

$$-\frac{\delta^i_j}{n+1}\dot\partial_{h]}\big(\dot\partial_p B^{pm}\big)\Big\} + \frac{V_i n\delta^i_{[k}}{n^2-1}2\Big\{\sigma_{m(h)]}\big(\dot\partial_j\dot\partial_p B^{pm}\big) - \sigma_{m(p)}\big(\dot\partial_j\dot\partial_{h]}B^{pm}\big)\Big\}$$

$$+V_i 2\sigma_m\sigma_r\Big[\dot\partial_j\big(\dot\partial_{[k}B^{sm}\big)\dot\partial_{h]}\dot\partial_s B^{ir} + \frac{\dot x^i}{n+1}\Big\{\dot\partial_j\dot\partial_{[k}\big(\dot\partial_{h]}B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr}$$

$$-\dot\partial_j\dot\partial_{[k}\big(\dot\partial_p B^{sm}\big)\dot\partial_j\dot\partial_{[k}\big(\dot\partial_p B^{sm}\big)\dot\partial_j\dot\partial_{[k}\big(\dot\partial_p B^{sm}\big)\dot\partial_{h]}\dot\partial_s B^{pr} + \dot\partial_{[k}\big(\dot\partial_{h]}B^{sm}\big)\dot\partial_j\big(\dot\partial_p\dot\partial_s B^{pr}\big)$$

$$-\dot\partial_{[k}\big(\dot\partial_p B^{sm}\big)\dot\partial_j\big(\dot\partial_{h]}\dot\partial_s B^{pr}\big) - \dot\partial_j\big(\dot\partial_{[k}B^{sm}\big)\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr} - \dot\partial_j\big(\dot\partial_p B^{sm}\big)\dot\partial_{[k}\big(\dot\partial_{h]}\dot\partial_s B^{pr}$$

$$-\big(\dot\partial_{[k}B^{sm}\big)\dot\partial_j\dot\partial_{h]}\big(\dot\partial_p\dot\partial_s B^{pr}\big) - \big(\dot\partial_p B^{sm}\big)\dot\partial_j\dot\partial_{[k}\big(\dot\partial_{h]}\dot\partial_s B^{pr}\big)\Big\} - \frac{\delta^i_j}{n+1}\Big\{\dot\partial_p\big(\dot\partial_{[h}B^{sm}\big)\dot\partial_{k]}\dot\partial_s B^{pr}\Big\}$$

$$+\frac{n\delta^i_{[k}}{n^2-1}\Big\{\big(\dot\partial_j\dot\partial_{h]}B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr} - \big(\dot\partial_j\dot\partial_p B^{sm}\big)\dot\partial_{h]}\dot\partial_s B^{pr} + \big(\dot\partial_{h]}B^{sm}\big)\dot\partial_j\dot\partial_p\dot\partial_s B^{pr}$$

$$-\big(\dot\partial_p B^{sn}\ \dot\partial_j\dot\partial_{h]}\dot\partial_s B^{pr}\Big\} + \frac{\delta^i_{[k}}{n^2-1}\Big\{\dot\partial_{h]}\big(\dot\partial_j B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr} - \dot\partial_{h]}\big(\dot\partial_p B^{sm}\big)\dot\partial_j\dot\partial_s B^{pr}$$

$$-\big(\dot\partial_j B^{sm}\big)\dot\partial_{h]}\dot\partial_p\dot\partial_s B^{pr} - \big(\dot\partial_p B^{sm}\big)\dot\partial_{[h}\dot\partial_j\dot\partial_s B^{pr}\Big\}\Big] + \frac{V_i\dot x^l\delta^i_{[k}}{n^2-1}\Big[\dot\partial_j\big(\dot\partial_{h]}\dot\partial_l B^{sm}\big)\dot\partial_p\dot\partial_s B^{pr}$$

$$-\dot\partial_j\dot\partial_{h]}\big(\dot\partial_p B^{sm}\big)\dot\partial_l\dot\partial_s B^{pr} + \dot\partial_{h]}\big(\dot\partial_l B^{sm}\big)\dot\partial_j\dot\partial_p\dot\partial_s B^{pr} - \dot\partial_{h]}\big(\dot\partial_p B^{sm}\big)\dot\partial_j\dot\partial_l\dot\partial_s B^{pr}$$

$$+\dot{\partial}_j\left(\dot{\partial}_l B^{sm}\right)\dot{\partial}_{h]}\left(\dot{\partial}_p\dot{\partial}_s B^{pr}\right)-\dot{\partial}_j\left(\dot{\partial}_p B^{sm}\right)\dot{\partial}_{h]}\left(\dot{\partial}_l\dot{\partial}_s B^{pr}\right)+\left(\dot{\partial}_l B^{sm}\right)\dot{\partial}_j\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_s B^{pr}$$

$$-\left(\dot{\partial}_p B^{sm}\right)\dot{\partial}_j\dot{\partial}_{h]}\dot{\partial}_l\dot{\partial}_s B^{pr}\Big\}\Big]-V_i 2\sigma_{m(p)}\left\{\frac{\dot{x}^i}{n+1}\dot{\partial}_j\left(\dot{\partial}_{[k}\dot{\partial}_{h]}B^{pm}\right)+\frac{\delta^i_{[k}}{n^2-1}\dot{\partial}_{h]}\dot{\partial}_l B^{pm}\right.$$

$$\left.+\frac{\dot{x}^l\delta^i_{[k}}{n^2-1}\dot{\partial}_j\left(\dot{\partial}_{h]}\dot{\partial}_l B^{pm}\right)\right\}+\frac{V_i\delta^i_{[k}}{n^2-1}2\left\{\sigma_{m(j)}\dot{\partial}_{h]}\left(\dot{\partial}_p B^{pm}\right)\right\}+\frac{V_i\dot{x}^l\delta^i_{[k}}{n^2k-1}2\left\{\sigma_{m(l)}\dot{\partial}_j\left(\dot{\partial}_{h]}\dot{\partial}_p B^{pm}\right)\right\}$$

...(29)

which represents the conformal transformation of the decomposition tensor under the change (1).

Thus we state

Theorem 5. Under the decomposition (20), the conformal decomposition tensor ϕ_{jkh} is expressed in the form (29).

Transvecting equation (20) by $\dot{x}^j$, we have

$$W^i_{kh}=X^i\phi_{kh} \tag{30}$$

where

$$\bar{\phi}_{kh}=\phi_{jkh}\dot{x}^j \tag{31}$$

Using the equation (23), (28) and (30) in the equation (12), we obtain

$$\bar{\phi}_{kh}=e^\sigma V_i\left(W^i_{kh}\right)-e^\sigma V_i 2\sigma_m\left[\left(\dot{\partial}_{[k}B^{im}\right)_{(h)]}-\frac{\dot{x}^i}{n+1}\left\{\left(\dot{\partial}_p\dot{\partial}_{[k}B^{pm}\right)_{(h)]}+\left(\dot{\partial}_{[k}B^{rm}\right)G^p_{h]rp}\right\}\right.$$

$$+\frac{1}{n^2-1}\delta^i_{[k}\left\{(n+1)\left(\dot{\partial}_{h]}B^{pm}\right)_{(p)}-n\left(\dot{\partial}_p B^{pm}\right)_{(h)]}-\left(\dot{\partial}_{h]}\dot{\partial}_p B^{pm}\right)_{(r)}\dot{x}^r+2B^{rm}G^p_{h]rp}\right\}\Big]$$

$$+e^\sigma V_i 2\sigma_{m[(k)}\left\{\dot{\partial}_{h]}B^{im}-\frac{n}{n^2-1}\delta^i_{h]}\dot{\partial}_p B^{pm}-\frac{\dot{x}^i}{n+1}\dot{\partial}_{h]}\dot{\partial}_p B^{pm}\right\}$$

$$+\frac{e^\sigma V_i 2}{n^2-1}\sigma_{m(p)}\delta^i_{[k}\left\{\dot{x}^p\dot{\partial}_{h]}\dot{\partial}_r B^{rm}-(n+1)\dot{\partial}_{h]}B^{pm}\right\}+e^\sigma V_i 2\sigma_m\sigma_r\left[\left(\dot{\partial}_{[k}B^{sm}\right)\left\{\dot{\partial}_{h]}\dot{\partial}_s B^{ir}\right.\right.$$

$$\left.-\frac{\dot{x}^i}{n+1}\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_s B^{pr}\right\}+\frac{1}{n-1}\delta^i_{[k}\left\{\left(\dot{\partial}_{h]}B^{sm}\right)\dot{\partial}_p\dot{\partial}_s B^{pr}-\left(\dot{\partial}_{h]}\dot{\partial}_s B^{pr}\right)\dot{\partial}_p B^{sm}\right.$$

$$\left.\left.+\frac{2}{n-1}B^{sm}\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_s B^{pr}\right\}\right]\Big] \tag{32}$$

Interchange the indices k and h in the equation (32), we get

$$\bar{\phi}_{kh}=-\bar{\phi}_{hk} \tag{33}$$

by vritue or relation $W^i_{kh} = -W^i_{hk}$ [4].

In the veiw of the equations (23) and (24), the equation (32), becomes

$$\bar{\phi}_{kh} = e^{\sigma}\phi_{kh} - e^{\sigma}V_i 2\sigma_m\left[\left(\dot{\partial}_{[k}B^{im}\right)_{(h)]} - \frac{\dot{x}^i}{n+1}\left\{\left(\dot{\partial}_p\dot{\partial}_{[k}B^{pm}\right)_{(h)]} + \left(\dot{\partial}_{[k}B^{rm}\right)G^p_{h]rp}\right\}\right.$$

$$+\frac{1}{n^2-1}\delta^i_{[k}\left\{(n+1)\left(\dot{\partial}_{h]}B^{pm}\right)_{(p)} - n\left(\dot{\partial}_p B^{pm}\right)_{(h)]} - \left(\dot{\partial}_{h]}\dot{\partial}_p B^{pm}\right)_{(r)}\dot{x}^r + 2B^{rm}G^p_{h]rp}\right\}\Bigg]$$

$$+e^{\sigma}V_i 2\sigma_{m|(k)}\left\{\dot{\partial}_{h]}B^{im} - \frac{n}{n^2-1}\delta^i_{h]}\dot{\partial}_p B^{pm} - \frac{\dot{x}^i}{n+1}\dot{\partial}_{h]}\dot{\partial}_p B^{pm}\right\}$$

$$+\frac{e^{\sigma}V_i 2}{n^2-1}\sigma_{m(p)}\delta^i_{[k}\left\{\dot{x}^p\dot{\partial}_{h]}\dot{\partial}_r B^{rm} - (n+1)\dot{\partial}_{h]}B^{pm}\right\} + e^{\sigma}V_i 2\sigma_m\sigma_r\left[\left(\dot{\partial}_{[k}B^{sm}\left\{\dot{\partial}_{h]}\dot{\partial}_s B^{ir}\right.\right.\right.$$

$$-\frac{\dot{x}^i}{n+1}\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_s B^{pr}\right\}\frac{1}{n-1}\delta^i_{[k}\left\{\left(\dot{\partial}_{h]}B^{sm}\right)\dot{\partial}_p\dot{\partial}_s B^{pr} - \left(\dot{\partial}_{h]}\dot{\partial}_s B^{pr}\right)\dot{\partial}_p B^{sm}\right.$$

$$\left.\left.+\frac{2}{n-1}B^{sm}\dot{\partial}_{h]}\dot{\partial}_p\dot{\partial}_s B^{pr}\right\}\right] \tag{34}$$

Which gives the conformal transformation of decomposition tensor $\bar{\phi}_{kh}$ under the conformal change.

Theorem 6. Under the decomposition (20) and (30) the conformal decomposition tensor $\bar{\phi}_{kh}$ is expressed in the form (34).

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EXTENSION OF *P*-TRNSFORM TO A CLASS OF GENERALISED FUNCTION

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ABSTRACT

P-transform is a new transform, which is defined on $0<t<\infty$. Testing function space P_β has been defined, so that the kernal function of the transform belongs to P_β. Properties of P_β have been studied. *P*-transform has been extended to a class of generalized function. *P*-transform has been shown as a particular case of convolution transform. A real inversion formula has been derived for *P*-transform.

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1. Introduction. Let us consider a transform

$$(1.1)\qquad P[f(t)] = F(s) = \frac{2}{\pi}\int_0^\infty \frac{t^5}{t^6+s^6} f(t)dt\ (0<\mathrm{Re}(s)<\infty),$$

where *f*(*t*) is a suitably restricted conventional function defined on the positive real line $0<t<\infty$ and $0<\mathrm{Re}\, s<\infty$. The above transform maps *f*(*t*) into a complex valued function F(s).

Testing Function space P_β and its Dual P'_β.

Let P_β be the space of all complex valued smooth (infinitely differentiable) fucction $\phi(t)$ defind on the positive real line $0<t<\infty$ s.t. for each $f(t)\in P_\beta$.
Then

$$(1.2)\qquad \rho_k(\phi) = \sup_{0<t<\infty}\left|(1+t)^\beta D^k\phi(t)\right| (\beta\le 1) \qquad (k=0,1,2\ldots).$$

Hence, ρ_k is a norm on $P_\beta\,(k=0,1,2\ldots)$ and $\{\rho_k\}_{k=0}^\infty$ is a multinorm on P_β.

Thus, P_β is a countably multinormed space Zemanian ([3], pp 8-9). Here P-Transform is a new transform defined on $0<t<\infty$. Testing function space P_β has been defined so that the kernel function of the transform belongs to P_β.

In the present paper, we study the properties of P_β and extend P-transform to a class of generalized function. P-transform is shown as a particular case of convolution transform. Finally a real inversion formula is derived for P-transform.

2. Lemmas. In this section, we establish four Lemmas.

Lemma 1. P_β is a complete space.

Proof. To prove that P_β is a complete space. It is sufficient to prove that every Cauchy sequence in P_β is a convergent sequence in P_β.

Therefore, for every $m,n>M_k$ (a fixed positive enteger)

$\exists$ a small $+ve\ \epsilon$ s.t. $\rho_k|\phi_n-\phi_m|<\varepsilon$.

Consequently, we get

$$(2.1)\quad \left|(1+t)^\beta D^k\phi_n(t)-\phi_m(t)\right|<\varepsilon \qquad (\beta\le 1).$$

But $\exists$ a smooth function $\phi(t)$ s.t. for each k and t, $D^k\phi_m(t)\to D^k\phi(t)$ as $m\to\infty$.

Due to Apostol [1,P, 402], we get

$$(2.2)\quad \left|(1+t)^\beta D^k\phi_n(t)-\phi(t)\right|<\varepsilon. \qquad (0<t<\infty, n>M_k).$$

Therefore, as $n\to\infty$

$\rho_k|\phi_n-\phi|\to 0$ for each k ($k=0,1,2\ldots$).

Also, since $\phi_n(t)\in P_\beta$, so we get

$$(2.3)\quad \left|(1+t)^\beta D^k\phi_n(t)\right|<C_k,$$

where C_k is a constant not dependindg upon n. An appeal to (2.2) and (2.3) gives,

$$\left|(1+t)^\beta D^k\phi(t)\right|=\left|(1+t)^\beta D^k\phi(t)-\phi_n(t)+\phi_n(t)\right|$$

$$\le\left|(1+t)^\beta D^k\phi(t)-\phi_n(t)\right|+\left|(1+t)^\beta D^k\phi_n(t)\right|<\varepsilon+C_k \text{ (from 2.2 and 2.3)},$$

which shows that the limit function $\phi(t)\in P_\beta$.

Hence, $\{\phi_n\}$ is a convergent sequence in P_β.

Therefore, P_β is a complete countably multinormed space.

Let P'_β be the dual of P_β. Then $f \in P'_\beta$ iff it is a continuous and linear functional on P_β since P_β is a testing function space (Zemanian [3],pp 38, 391).

So, P'_β is the space of generalised functions.

Thus, for any $f \in P'_\beta$ and $\phi \in P_\beta$, value of the generalised function is denoted by $< f, \phi >$.

Lemma 2. To prove $(2/\pi)\ t^5/(t^6+S^6) \in P_\beta$ for $\beta \le 1$, $0 < t < \infty$ and $0 < \mathrm{Re}\, s < \infty$.

Proof. Let us consider,

$$2/\pi.5t^4\left(s^6+t^6\right)-6t^{10}/(t^6+s^6)^2.$$

Consequently,we get

$(1+t)^\beta D^k(2/\pi)t^5/(t^6+s^6) = P_k(t_1)/Q_k(t)\,(\beta \le 1)$,

where $P_k(t)$ and $Q_k(t)$ are the polynomials in t s.t. the order of $Q_k(t)$>order of $P_k(t)\ \forall\ k = 0,1,2\ldots$.

Therefore, we get :

$\sup\limits_{0<t<\infty}\left(\left|(1+t)^\beta D^k\right|(2/\pi)t^5/\left(s^6+t^6\right)\right)$ bounded for all $\beta \le 1, 0 < s < \infty$ and $k = 0,1,2\ldots$

Hence, $(2/\pi)\ t^5/\left(t^6+s^6\right) \quad \in P_{\beta^-}$.

Lemma 3. $D(I)$ is a subspace of $P_{\beta'}$, where $D(I)$ has been defined [3, pp. 8-9].

Proof. Let. $\phi \in D(I) \Rightarrow \sup\limits_{t\in I}\left|D^k\phi(t)\right|$ is bounded,

where $f(t)$ is a complex valued smooth function non zero within the compact set K of $I =]0, \infty[$ and zero outside K.

$\Rightarrow \quad \sup\limits_{0<t<\infty}\left|(1+t)^\beta D^k\phi(t)\right|$ is bounded for all $\beta \le 1, k = 0,1,\ldots$

$\Rightarrow \quad \phi \in P_\beta$.

Therefore, we get

(2.4) $\quad D(I) \le P_\beta$.

Thus, the convergence of a sequence in $D(I)$ implies the convergence of the sequence in $P_{\beta'}$ Consequently the restriction of P_β to $D(I)$ is in $D(I)$.

However, $D(I)$ is not dense in P_β. Thus, we cannot identify P'_β with a subspace of $D'(I)$. Actually, the different members of P'_β can be found whose restriction to $D'(I)$ are identical.

Lemma 4. P_β is a dense subspace of $E(I)$ where $E(I)$ has been defined [3, pp. 8-9]

Proof. Let $\phi \in P_\beta \Rightarrow \sup_{0<t<\infty}\left|(1+t)^\beta D^k\phi(t)\right|$ is bounded, where $\beta \le 1$, $k = 0,1,2...$

$\Rightarrow \sup_{t\in k}\left|D^k\phi(t)\right|$ is bounded where k is a compact set of $I=[0, \infty]$

$\Rightarrow \phi \in E(I)$.

Therefore, we get

(2,5) $\quad P_\beta \subseteq E(I)$.

An appeal to (2.4) and (2.5) gives

(2.6) $D(I) \subseteq P_\beta \subseteq E(I)$.

Also, $D(I)$ is a dense subspace of $E(I)$. [Zemainan (3.P.3.7].

Thus, from (2.6) it follows that P_β is a dense subspace of $E(I)$. Hence, we get the result.

3. Extension of the *P*-Transform to a Class of Generalised Function.

Let us call f as a P-transformable generalised function if it possesses the following properties :

i. f is a functional on some domain $d(f)$ of conventional functions.

ii. f is additive in the sense that if $\theta,\phi,\theta+\phi$ are all members of $d(f)$, then $\langle f,\theta+\phi\rangle = \langle f,\theta\rangle + \langle f,\phi\rangle$.

iii. $P'_\beta \subseteq d(f)$ and the restriction of f to P_β is in P'_β.

Also $2/\pi.\ t^5/\left(t^6+s^6\right) \in P_\beta$ for $\beta \le 1; 0 < \operatorname{Re} s < \infty$.

We define the generalised P-transform of f by

(3.1) $F(s) = P[f(t)] = \langle f(t),\ (2/\pi).t^5(t^6+s^6)\rangle$

where, $s \in \Omega f$; and

(3.2) $\quad \Omega f = \{s : 0 < \operatorname{Re}(s) < \infty\}$.

Thus, f is called the region of definition of P-transform and $0, \infty$ are the abscissa

of definition. Morever, we call to the operation $P{:}f \to F$ as P-Transform.

4. P-transform as a Particular Case of Convolution Transform.

Let us consider the convolutional transform

$$(4.1) \qquad H(x) = \int_{-\infty}^{\infty} h(y)\, G(x-y)\, dy (-\infty < x < \infty)$$

and let us choose for the kernel

$$(4.2) \qquad G(x-y) = (2/\pi).\ 1/\left(1 + e^{6(X-y)}\right)$$

and $\quad G(t) = (2/\pi)/(1+e^{6t})$.

Let us change the variables of (4.1) and (4.2) by putting $s=e^x$ and $t=e^y$. Thus we get

$$(4.3) \qquad H(\log s) = \frac{2}{\pi} \int_{-\infty}^{\infty} h(\log t)\ t^5/(t^6+s^6)\ dt$$

$$= \int_{-\infty}^{\infty} h(\log t)\ t^5 / \left(t^6 + s^6\right)\ dt \quad (0 < \mathrm{Re}\,(s) < \infty).$$

If we put $H(\log s) = F(s)$ and $h(\log t) = f(t)$, then (4,3) becomes

$$(4.4) \qquad F(s) = \frac{2}{\pi} \int_{-\infty}^{\infty} f(t)\ t^5/(t^6+s^6)\, dt (\, 0 < \mathrm{Re}(s) < \infty),$$

which is identical with (1.1).

Therefore, (1.1) is a particular case of convolution Transform.

5. A Real Inversion Formula for p-Transform.

Let $1/E(s) \int_{-\infty}^{\infty} = G(t) e^{-st} dt$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+e^{6t}} \quad e^{-st} dt$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{1}{1+x^{-6}.} x^{s-1} dx \text{ (by substituting } e^{-t} = x).$$

Therefore, we get $\dfrac{1}{E(s)} = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} \frac{x^{s-1}}{\left(1+1/x^6\right)} dx$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{x^5 x^s}{x^6+1} dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{x^{s+5}}{1+x^6} dx$$

$$(5.1) \quad \frac{1}{E(s)} = \frac{1}{3Sin[(s+6)/6]\pi.} \Rightarrow E(s) = 3\sin\left(\frac{S+6}{6}\right)\pi$$

(i) if (s+5) is an even positive Integer.

(ii) (s+5) <6 i.e. s <1.

The result (5.1) is obtained by the contour integration of $\frac{z^{s+5}}{1+z^6}$ on a semi circle with centre at the origin and radius $R \to \infty$. Therefore, $E(s)=3\text{Sin}[(s+6)/6]\pi$. where $E(s)$ is called the inversion function corresponding to $G(t)$.

Now the inversion formula of (1.1) is given by {2p.8]

$3\ \text{Sin}\ [(D+6)/6]\ \pi\ H(t)=h(t)\ (-\infty < t < \infty)$.

The substitution $t = \log s \Rightarrow H(\log s) = F(s) \Rightarrow h(\log s) = f(s)$.

Therefore, (5.1) gives

$$(5.2) \quad 3\sin[(D+6)/6]\pi F(s) = f(s) \qquad (0 < s < \infty)$$

which in an inversion formula of (1.1).

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ON APPROXIMATION OF A FUNCTION BY GENERALIZED NÖRLUND MEANS OF ITS FOURIER SERIES

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ABSTRACT

The aim of this paper is to establish a theorem on approximation of a function by generalized Nörlund means of its Fourier series which generalize several previous results.

Keywords and Phrases : Fourier series, Generalized Nörlund means.
2000 Mathematics Subject Classification : Primary 42A24; Secondary 42B08.

1. Introduction. In 1943, Iyenger [7] proved a theorem on harmonic summability of Fourier series: The result of Iyenger [7] was generalized by Hardy [4], Hirokawa [5], Hirokawa and Kayashima [6], Pati [10], Prasad [14], Pandey [11], Rajagopal [15], Siddiqui [16] and Singh [17], for (N,p_n) summabiliity of Fourier series under different conditions. Dealing with Cesáro-means of Fourier series of a function, Flett [2] has obtained a result on the degree of approximation. This result was generalized by Izumi, Satô and Sunouchi [8] and Siddiqui [16] by using Nörlund means. Working in the same direction, the result of Siddiqui [16] has been extended by Porwal [12], Gupta and Pandey [3] and Chourasia [1]. The purpose of this paper is to establish a very general result than those of Porwal [12], Gupta & Pandey [3] and Chourasia [1] so that their results come out as particular cases.

2. Definitions and Notations. Let f be 2π-periodic and integrable in the Lebesgue sense. The Fourier series associated with f at a point x is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where a_n and b_n are Fourier coefficients of f and are determined by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \qquad (n \geq 0)$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \qquad (n>0).$$

Let (p_n) and (q_n) be sequences of positive constants such that

$$P_n = \sum_{k=0}^{n} p_k, \quad Q_n = \sum_{k=0}^{n} q_k, \quad R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0 \qquad (n \geq 0)$$

where P_n, Q_n and $R_n \to \infty$ as $n \to \infty$.

Let $\sum_{n=0}^{\infty} a_n$ be series whose n^{th} partial sum is denoted by S_n.

Write

$$N_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_k q_{n-k} S_{n-k}.$$

If $N_n^{p,q} \to s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} a_n$ or the sequence (S_n) is said to be summable to S by generalized Nörlund method. We write for each real x,

$$\phi(t) = f(x+t) + f(x-t) - 2f(x)$$

and τ or $[1/t]$, the integral part of $1/t$ in $0 < t \leq \pi$.

3. Main Theorem. If $0 \leq \alpha \leq 1$, $0 < \delta \leq \pi$ and x is a point :

$$\Phi(t) = \int_t^{\delta} |\phi(u)| \frac{R_{(1/u)}}{u}\, du$$

$$= O\left[\left\{R_{(1/u)} h(t)\right\}^{\alpha}\right] \text{ as } t \to 0, \qquad \ldots(3.1)$$

where

$$\left\{R_{(1/u)} h(t)\right\}^{\alpha} \to \infty \text{ as } t \to 0, \qquad \ldots(3.2)$$

$$\int_0^t \left\{R_{[1/t]} h(u)\right\}^{\alpha} du = O\left[t\left\{R_{[1/t]} h(t)\right\}^{\alpha}\right] \qquad \ldots(3.3)$$

and $h(t)$ is a positive increasing function such that

$h(t) \to 0$ as $t \to 0$, then

$$N_n^{p,q}(f;x) - f(x) = O\left[(R_n)^{\alpha-1} h_{[1/n]}^{\alpha}\right] + O\left[1/R_n\right].$$

4. Lemmas. We shall require the following lemmas in the proof of the theorem:

Lemma 1. Let the sequence (p_n) be non-negative and non-decreasing, then for $0 \le a < b < \infty$ and for any n,

$$\left| \sum_{k=a}^{b} p_{n-k} \exp(ikt) \right| \le MP_\tau$$

uniformly in $0 < t \le \pi$, where M is some positive constant not necessarily the same at each occurrence.

For its proof see Mc Fadden [9].

Lemma 2. Let the sequences (p_n) and (q_n) be non-decreasing, then for uniformly in $0<t<\pi$,

$$\left| \sum_{k=a}^{b} p_n q_{n-k} \sin(n-k+1/2)t \right| = O(R_\tau).$$

Proof. We have

$$\left| \sum_{k=a}^{n} p_n q_{n-k} \sin(n-k+1/2)t \right| = \left| \sum_{k=0}^{\tau-1} + \sum_{k=\tau}^{n} \right| \le \sum{}_1 + \sum{}_2$$

where

$$\sum{}_1 = \left| \sum_{k=0}^{\tau-1} p_n q_{n-k} \sin(n-k+1/2)t \right| \le \sum_{k=0}^{\tau-1} p_n q_{n-k} \le \sum_{k=0}^{\tau} p_n q_{n-k}.$$

Now, since $q_n \ge q_{n+1}$, therefore

$$q_{\tau-k} \ge q_{n-k} \text{ for } n \ge \tau.$$

Hence

$$\sum{}_1 \le \sum_{k=0}^{\tau} p_k q_{\tau-k}.$$

By Abel's lemma

$$\sum{}_2 \le p_\tau \max_{\tau \le r \le n} \left| \sum_{k=\tau}^{r} q_{n-k} \sin(n-k+1/2)t \right|$$

where

$$\left| \sum_{k=\tau}^{r} q_{n-k} \sin(n-k+1/2)t \right|$$

$$= \left| \sum_{k=\tau}^{r} q_{n-k} \left\{ \sin\left(n-k+\tfrac{1}{2}\right)t \cos kt - \cos(n+1/2)t \sin kt \right\} \right|$$

$$\le \left|\sum_{k=\tau}^{r} q_{n-k}\cos kt\right| + \left|\sum_{k=\tau}^{r} q_{n-k}\sin kt\right|$$

$\le MQ_\tau$, by lemma 1.

Thus $\sum_2 \le MP_\tau Q_\tau \le MR_\tau$

since $R_\tau = \sum_{k=0}^{\tau} p_k q_{\tau-k} \ge p_\tau \sum_{k=0}^{\tau} q_{\tau-k} = P_\tau Q_\tau$.

Combining above results we finally have

$$\left|\sum_{k=0}^{n} p_k q_{n-k}\sin(n-k+1/2)t\right| = O(R_\tau).$$

Lemma 3. Let (p_n) and (q_n) be the sequences as in lemma 2. Then

$$\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{\sin(n-k+1/2)}{\sin t/2} = \begin{cases} O(n) & ; \ (O \le t \le 1/n) \\ O(R_\tau/tR_n) & ; \ (1/n < t < \delta) \\ O(1/R_n) & ; \ (\delta \le t \le \pi) \end{cases}.$$

Proof. We have for $0 \le t \le 1/n$

$$\left|\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{\sin(n-k+1/2)t}{\sin t/2}\right| = O\left|1/R_n \sum_{k=0}^{n} p_k q_{n-k}\frac{(n-k+1/2)t}{t/2}\right|$$

$$= O(n) \text{ for } 0 \le t \le 1/n.$$

Now for $1/n < t < \delta$

$$\left|\frac{1}{2\pi R_n}\sum_{k=0}^{n} p_k q_{n-k}\frac{\sin(n-k+1\cdot 2)t}{\sin t/2}\right|$$

$$\le \frac{1}{2\pi R_n}\left|\sum_{k=0}^{n} p_k q_{n-k}\frac{\sin(n-k+1/2)t}{\sin t/2}\right|$$

$= 1/R_n\, O(R_\tau/t)$, by lemma (2) and since $\sin t/2 \ge t/\pi \quad (0 \le t \le \pi)$

$= O(R_\tau/tR_n)$.

Similarly we can prove the third part of the lemma.

Lemma 4. Under the condition (3.1), we have

$$\int_0^t |\phi(u)|du = O\left[t\left(R_{[1/t]}\right)^{\alpha-1}(h(t))^{\alpha}\right] \text{ as } t \to 0$$

Proof. We write

$$\Phi(t) = \int_t^{\delta} \frac{|\phi(u)|}{u} R_{[1/u]} du$$

then by hypothesis of theorem, we obtain

$$\int_0^t |\phi(u)| R_{[1/u]} du = -\int_0^t u \Phi(u) du$$

$$= -[u\phi(u)]_0^t + \int_0^t \Phi(u) du$$

$$= -t\Phi(t) + \int_0^t \Phi(u) du$$

$$= O\left[t\left(R_{[1/t]} h(t)\right)^{\alpha}\right] + \int_0^t \left(R_{[1/u]} h(u)\right)^{\alpha} du$$

$$= O\left[t\left(R_{[1/t]} h(t)\right)^{\alpha}\right], \text{ by (3.2)} \qquad (4.1)$$

Hence, we have

$$\int_0^t |\phi(u)| du = \int_0^t |\phi(u)| \frac{R_{[1/u]} du}{R_{[1/u]}}$$

$$= O\left[\frac{1}{R_{[1/t]}} \int_0^t |\phi(u)| R_{[1/t]} du\right]$$

$$= O\left[\frac{1}{R_{[1/t]}} \left\{ t R_{[1/t]}^{\alpha} (h(t))^{\alpha} \right\}\right], \qquad \text{by (4.1)}$$

$$= O\left[t R_{[1/t]}^{\alpha-1} (h(t))^{\alpha}\right] \quad \text{as } t \to 0.$$

5. Proof of the main theorem. Let $S_n(f;x)$ be the n^{th} partial sums of the Fourier series of *f*. Then (N, p_n, q_n) means of $S_n(f;x)$ is given by

$$N_n^{p,q}(f;x) = \frac{1}{R_n} \sum_{k=0}^{n} p_k q_{n-k} S_{n-k}(f;x)$$

where

$$S_n(f;x) = \frac{1}{\pi} \int_0^{\pi} \{f(x+t) + f(x-t)\} \frac{\sin(n+1/2)t}{2\sin t/2} dt.$$

Hence

$$N_n^{p,q}(f;x)-f(x)=\frac{1}{R_n}\sum_{k=0}^{n}p_k q_{n-k}\{S_{n-k}(f;x)-f(x)\}$$

$$=\frac{1}{2\pi R_n}\int_0^\pi\frac{\phi(t)}{\sin t/2}\sum_{k=0}^{n}p_k q_{n-k}\sin(n-k+1/2)\,tdt$$

$$=\int_0^{1/n}+\int_{1/n}^{\delta}+\int_{\delta}^{\pi}$$

$$=I_1+I_2+I_3 \text{ (say)}. \tag{5.1}$$

Now $|I_1|=O\left(n\int_0^{1/n}\phi(t)\,dt\right)$, by lemma 3

$$=O\left(n\frac{1}{n}(R_n)^{\alpha-1}\left(h_{[1/n]}\right)^{\alpha}\right), \text{ by lemma 4}$$

$$=O\left((R_n)^{\alpha-1}\left(h_{[1/n]}\right)^{\alpha}\right). \tag{5.2}$$

Next, we have

$$|I_2|=O\left[\frac{1}{R_n}\int_{1/n}^{\delta}\frac{|\phi(t)|}{t}R_{[1/t]}dt\right], \text{ by lemma 3}$$

$$=O\left(\frac{1}{R_n}\left(R_n h_{[1/n]}\right)^{\alpha-1}\right), \quad \text{by (1.1)}$$

$$=O\left((R_n)^{\alpha-1}\left(h_{[1/n]}\right)^{\alpha}\right). \tag{5.3}$$

Finally, we consider

$$|I_3|=O\left[\frac{1}{R_n}\int_{\delta}^{\pi}\frac{|\phi(t)|}{t}R_{[1/t]}dt\right], \text{ by Lemma 3}$$

$$=O(1/R_n) \tag{5.4}$$

by Reimann Lebesgue theorem and regular conditions of summation procees. Combining (5.1), (5.2), (5.3) and (5.4) we get the proof of theorem.

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A COMMON FIXED POINT THEOREM IN MENGER SPACES

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ABSTRACT

In the present paper we prove a common fixed point theorem for six mappings in Menger spaces. Our result unifies and generalizes some of the previous known theorems due to Dimri and Chandola [2], Sharma [9], Singh and Chauhan [11] by using a different contraction condition. To some extent we replace condition of compatibility by R-weak commutativity of the mappings.

AMS Subject Classification (2000) :$47H10$, $54H25$

Keywords : Complete Menger space, R-weakly commuting maps, Common fixed point.

1. Introduction and Preliminaries. Menger [4] introduced the notion of probabilistic metric spaces (PM-spaces). Sehgal and Bharucha-Reid [7] initiated the study of fixed points in a subclass of probabilstic metric spaces. They extended the notion contraction and local contractions to the setting of Menger spaces. The study of this space expanded rapidly with the pioneering works of Schweizer and Sklar[6].

Sessa [8] initiated the tradition of improving commutativity conditions in metrical common fixed point theorems and introduced the notion of weak commutativity. Motivated by Sessa [8], Jungck [3] introduced the notion of compatible mappings. In this connection Pant [5] introduced the notion of R-weak commutativity which asserts that a pair of self-mappings (f,g) on a metric space (X,d) is said to be R-weakly commuting if there exists $R>0$ such that

$$d(fgx,gfx) \leq R\, d(fx,gx) \text{ for all } x \in X.$$

Recently, Singh and Tomar [12] presented a brief development of weaker forms of commuting maps and obtain some results for non-commuting and non-continuous maps on non-complete metric spaces. In 1998, Dimri and Gairola [1] introduced the concept of R-weak commutativity for a pair of maps in probabilistic metric spaces and established a fixed point theorem for generalized non-linear contraction.

In the present paper we prove a common fixed point theorem for six mappings in Menger spaces by using the notion of R-weak commutativity. Our result unifies and generalizes some results due to Dimri and Chandola [2], Sharma [9], Singh

and Chauhan [11] with less restrictive conditions on mappings.

Definition 1.1 [7]. A distribution function is a mapping $F: R \to R^+$ which is non-decreasing and left continuous with inf $F=0$ and sup $F=1$. We shall denote D by the set of all distribution functions.

Definition 1.2 [7]. A probabilistic metric space is an ordered pair (X,F), where X is an abstract set and F is a mapping of $X \times X$ into D i.e., F associates a distribution function $F(p,q)$ with every pair (p,q) of points in X. We shall denote the distribution function $F(p,q)$ by $F_{p,q}$. The functions $F_{p,q}$ are assumed to satisfy the following conditions:

(PM-1) $F_{p,q}(x) = 1$ for all $x > 0$ iff $p=q$,

(PM-2) $F_{p,q}(0) = 0$

(PM-3) $F_{p,q} = F_{q,p}$,

(PM-4) If $F_{p,q}(x)=1$ and $F_{q,r}(y)=1$ then $F_{p,r}(x+y)=1$.

Definition 1.3 [7]. A triangular norm (briefly, a t -norm) is a function $t:[0,1] \times [0,1] \to [0,1]$ satisfying the following conditions:

(i) $t(a,1)=a$ and $t(0,0)=0$,

(ii) $t(a,b)=t(b,a)$,

(iii) $t(c,d) \geq t(a,b)$ for $c\ a, \geq d \geq b$,

(iv) $t(t(a,b),c)=t(a,t(b,c))$.

Definition 1.4 [6]. A Menger PM-space is triplet (X,F,t) where (X,F) is a PM-space and t-norm t satisfies

$$F_{P,r}(x,y)\ t\ \{F_{p,q}(x),F_{q,r}(y)\} \text{ for all } x,y \geq 0 \text{ and } p,q,r \in X.$$

Note that among a number of possible choices for $t, t(a,b) = \min\{a,b\}$ or simply "t=min" is the strongest possible universal t (cf.[6]).

Schweizer and Sklar [6] have proved that if (X,F,t) is a Menger PM-space with a continuous t-norm, then X is a Hausdorff space in the topology T induced by the family of neighbourhoods

$$\{N_p(\varepsilon,\lambda): p \in X, \varepsilon > 0, \lambda > 0\} \text{ where } Np(\varepsilon,\lambda) = \{x \in X : F_{x,p}(\varepsilon) > 1-\lambda\}.$$

Defination 1.5 [1]. Let A and B be two mappings on probabilistic metric space X. The pair (A,B) will be called R-weakly commuting if and only if

$$F_{ABu,BAu}(Rx) \geq F_{Bu,Au}(x) \text{ for all } u \in X \text{ and } R>0.$$

Definition 1.6 [7]. A sequence$\{u_n\}$ in a probabalistic metric space X is said to converge to u if and only if for each $\lambda \geq 0$, $x \geq 0$, there exists a positive integer $N(x,\lambda) \in N$ such that

$$F_{u_n,u}(x) > 1-\lambda \text{ for all } n \geq N.$$

Or, equivalently $\lim_{n\to\infty} F_{u_n,u}(x) = 1$.

Lemma 1 [4]. Let $\{u_n\}$ be a sequence in a Menger space X. If there exists a number $k \in (0,1)$ such that

$F_{u_{n+2},u_{n+1}}(kx) \geq F_{u_{n+1},u_n}(x)$ for all $x>0$ and $n=1,2,\ldots$, then $\{u_n\}$ is a Cauchy sequence in X.

2. Main Result. In this section, we establish the following common fixed point.

Theorem : Let A,B,S,T,I and J be self mappings of a complete Menger space (X,F,t) where t is continuous and satisfies $t(x,x) \geq x$ for every $x\in[0,1]$, satisyfing the following conditions:

(1.1) $AB(X) \subset ST(X) \subset I(X)$

(1.2) if either (a) I or AB is continuous, pairs (AB,I) and (ST,J) are R-weakly commuting or (a') J or ST is continuous, pairs (AB, I) and (ST,J) are R-weakly commuting,

(1.3) $F_{ABu,STv}(kx)$ $F_{Iu,Jv}(x)$ for $0<k<1$, $x>0$, $u,v\in X$.

(1.4) (A,B) and (S,T) are commuting.

Then A,B,S,T, I and J have a unique common fixed point.

Proof: Let u_0 be an arbitrary point in X. Since $AB(\mathrm{X}) \subset J(\mathrm{X})$, we can find a point u_1 in X such that $ABu_0=Ju_1$. Also, since $ST(X)\subset I(X)$ we can choose a point u_2 with $STu_1=Iu_2$. Using this argument repeatedly one can construct a sequence $\{y_n\}$ such that

$$y_{2n}=ABu_{2n}=Ju_{2n+1} \text{ and}$$
$$y_{2n+1}=STu_{2n+1}=Iu_{2n+2} \text{ for } n=0,1,2,\ldots$$

Now from(1.3) and properties of t-norm,

$$\begin{aligned}
F_{y_{2n+1},y_{2n+2}}(kx) &= F_{STu_{2n+1},ABu_{2n+2}}(kx) \\
&= F_{ABu_{2n+2},STu_{2n+1}}(kx) \\
&\geq F_{Iu_{2n+2},Ju_{2n+1}}(x) \\
&= F_{y_{2n+1},y_{2n}}(x) \\
&= F_{y_{2n},y_{2n+1}}(x).
\end{aligned}$$

In general, $F_{y_n,y_{n+1}}(kx) \geq F_{y_{n-1},y_n}(x)$ for all $n \in N$.

Thus, by Lemma 1,$\{y_n\}$ is a Cauchy sequence in X, Since X is complete, $\{y_n\}$coverage to some point z in X.

Since $\{ABu_{2n}\}$, $\{Ju_{2n+1}\}$, $\{STu_{2n+1}\}$ and $\{Iu_{2n+2}\}$ are sub-sequences of $\{y_n\}$, they also converge to the point z as $n\to\infty$.

Case I. Let us assume that I is continuous. Then the sequences $\{I^2u_{2n}\}$ and $\{IABu_{2n}\}$ converge to Iz. Thus for $x>0,\lambda\in(0,1)$, there exists a positive integer $N(x,\lambda)$ such that

(1.5) $F_{IABu_{2n}}, Iz(x/2) > 1-\lambda$ and

$F_1^2 u_{2n}, Iz(x/2) > 1-\lambda$ for all $n \geq N(x,\lambda)$.

Using (1.5), we have

$$F_{ABIu_{2n},Iz}(x) > t\{F_{ABIu_{2n},IABu_{2n}}(x/2), F_{IABu_{2n},Iz}(x/2)\} \text{ for all } n \geq N.$$

Using R-weak commutativity of AB and I,we have

$$F_{ABIu_{2n},Iz}(x) > t\{F_{Iu_{2n},ABu_{2n}}(x/2R), F_{IABu_{2n},Iz}(x/2)\}.$$

Therefore from (1.5),

(1.6) $ABIu_{2n} \to Iz$ as $n \to \infty$.

From(1.3),

$$F_{ABIu_{2n},STu_{2n+1}}(kx) \geq F_{I^2u_{2n},Ju_{2n+1}}(x).$$

On Lettting $n \to \infty$ and from (1.6), we get

$F_{Iz,z}(kx) \geq F_{Iz,z}(x)$, which is not possible, since F is non-decreasing therefore $Iz=z$. Again using(1.3),

$$F_{ABz,STu_{2n+1}}(kx) \geq F_{Iz,Ju_{2n+1}}(x).$$

Letting $n \to \infty$, we have

$F_{ABz,z}(kx) \geq F_{Iz,z}(x) \to 1$ which implies that $ABz=z$.

Since $AB(X) \subset J(X)$, there exists a point w in X such that $JW=ABz=z$, so that $STz=STz=ST(Jw)$.

Now from (1.3),

$$F_{z,STw}(kx) = F_{ABz,STw}(kx)$$

$$\geq F_{Iz,Jw}(x)$$

$= F_{z,z}(x) \to 1$, a contradiction. Thus,

(1.7) $STw=z=Jw$, which shows that w is the coincidence point of ST and J.

Now, using the R-weak commutativity of (ST,J) and from (1.7), for $R>0$ we have

$$F_{STJw,JSTw}(Rx) \geq F_{STw,Jw}(x) \to 1.$$

Therefore $ST(Jw) = J(STw)$.

Hence $STz=ST(Jw)=JSTw=Jz$, which implies that z is also coincidence point of the pair (ST,J).

Using (1.3),

$F_{z,STz}(kx) = F_{ABz,STz}(kx)$ since ABz=z

$$\geq F_{Iz,Jz}(x)$$

implying thereby $STz=z=Jz$.

Therefore z is a common fixed point of AB, I, ST and J.

Case II. Now suppose that AB is continuous, so that the sequence $\{(AB)^2u_{2n}\}$ and

$\{ABIu_{2n}\}$ converge to ABz. Since (AB, I) are R-weakly commuting, therefore,

$$F_{IABu_{2n},ABz}(x) > t\{F_{IABu_{2n},ABIu_{2n}}(x/2), F_{ABIu_{2n},ABz}(x/2)\}$$

$$> t\{F_{ABu_{2n},Iu_{2n}}(x/2R), F_{ABIu_{2n},ABz}(x/2)\}$$

for all $n \geq N$.

Letting $n \to \infty$, above inequality implies that

(1.8) $\quad IABu_{2n} \to ABz$.

Again from (1.3), we have

$F_{ABz,z}(kx) \geq F_{AB,z}(x)$. Therefore $ABz=z$.

As earlier, there exists w in X such that ABz=z=Jw.

Then,

$$F_{(AB)^2u_{2n},STw}(kx) \geq F_{IABu_{2n},Jw}(x) ;$$

which on taking the limit $n \to \infty$ reduces to

$F_{z,STw}(kx) \geq F_{z,STw}(x)$, this implies that $STw=z=Jw$. Thus w is the coincidence point of (ST, J). Since the pair (ST, J) are R-weakly commuting. then $STz=Jz$.

Further, $F_{ABu_{2n},STz}(kx) \geq F_{Iu_{2n},Jz}(x)$ reduces to

$F_{z,STz}(kx) \geq F_{z,STz}(x)$ as $n \to \infty$, gives STz=z=Jz

Since $ST(X) \subset I(X)$,there exists a point in X such that

$Iy=STz=z$, then

$F_{ABy,z}(kx) = F_{A,By,STz}(kx) \geq F_{Iy,Jz}(x) = F_{z,z}(x) \to 1$ which gices $ABy=Z$.

Also (AB, I) are R-weakly commuting, we obtain

$$F_{ABz,Iz}(x) = F_{ABIy,IABy}(x) \text{ since } Iy=z=ABy$$

$$\geq F_{Iy,ABy}(x/R)$$

$$= F_{z,z}(x/R) \to 1 \text{ gives that}$$

$ABz=Iz=z$

Thus z is a common fixed point of AB, ST, I and J.

If the mapping ST or J is continous instead of AB or I, then the proof that z is a common fixed point of AB, ST, I and J is similar.

Let z' be another fixed point of I, J, AB and ST, then

$$F_{z,z'}(kx) = F_{ABz,STz'}(kx)$$

$$\geq F_{Iz,Jz'}(x)$$

$$\geq F_{z,z'}(x) \text{ implying thereby}$$

$z=z'$. Hence z is unique.

Finally, we prove that z is a unique common fixed point of A, B, S, T, I and J. We have shown that z is a unique common fixed point of A, B, S, T, I and J. Then, using commutativity of A and B, we have

$Az=A(ABz)=A(BAz)= AB(Az)$ which shows that Az is a fixed point of AB, but z is the unique fixed point of AB. Therefore

$$Az=z=ABz. \text{ Similarly } Bz=z=ABz.$$

Again using commutativity of S and T and in view of uniqueness of z, it can be shown that

$$Sz=z=STz, \qquad Tz=z=STz.$$

Hence z is a unique common fixed point of A, B, S, T, I and J.

Remark 1. If we put $I=J$, $A=S$ and $B=T=$Identity map in the **Theorem**, we get the following result:

Corollary1. Let A and J be self maps of a complete Menger space X such that (A,J) are R-weakly commuting, J is continous and

$$F_{Au,Av}(kx) \geq F_{Ju,Jv}(x) \text{ for all } u,v \in X,\ x>0 \text{ and } 0<k<1.$$

Then A and J have a unique common fixed point.

Remark 2. A number of fixed point theorems may be obtained for two to four mappings in metric, probabilistic and fuzzy metric spaces as the special cases from The **Theorem**.

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PERFORMANCE PREDICTION OF MIXED MULTICOMPONENTS MACHINING SYSTEM WITH BALKING, RENEGING, ADDITIONAL REPAIRMEN

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ABSTRACT

The present investigation deals with a stochastic model for multi components system with spares and state dependent rates. We study the machining system where failed units may balk with probability $(1-\beta)$ and renege according to exponential distribution. The queue size distribution in equilibrium state for the system having M operating units along with mixed spares of which S are warm and Y are cold, is established using product type method. Since the reliability of the system, depends upon the system configuration, the provision of r special additional repairmen which turn on according to a threshold rule depending upon the number of failed units in the system, is also made. The expressions for some performance measures are provided. The expressions for expected total cost per unit time has also been facilitated. By setting appropriate parameters some special cases are deduced which tally with earlier existing results.

Keywords : Machine repair, Mixed spares, Queue Size, Balking, Reneging, Additional repairmen, State dependent retes.

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1. Introduction. The study of fascinating area of machine repair problems via queueing theory approach can play a crucial role in predicting system descriptions of manufacturing and production systems. In the industrial world, the machine repair problems arise in many areas such as production system, computer network, communication system, distribution system, etc. When a

machine fails in a system, the interference occurs in the production if all spares have exhausted. A machine interference problem is said to be Markovian, if the inter arrival time and service time are both exponential. Machine repair problem with spares and additional repairmen is an extension of machine interference problem. In view of machine intérferenc, the provision of standby units is recommended as by using these units the system may keep working to provide the desired grade of service all the time. There are three types of standbys namely cold, warm and hot as defined below. A standby machine is said to be a cold standby when its failure rate is zero i.e. only operating units fail. In case of warm standby, the failure rate of spare unit is non zero but less than the failure rate of an operating machine, and is called hot standby when its failure rate is equal to an operating machine. The available standby unit may replace the failed unit whenever the operating unit fails. The behavior of the failed unit depends upon the number of failed units ahead of it.

For maintaining continuous magnitude of the production, it is recommended that the spare part support and additional removable repairman will be provided. It is worthwhile to have a glance on some of the relevant works done in this area. The queueing modeling of machine repair problem with spares and /or additional repairmen has been done by many researchers. Gross et al. (1977) considered the birth-death processes to study markovian finite population model with the provision of spare machines. Gupta (1997) introduced machine interference problem with warm spares, server vacations and exhaustive service. Jain (1998) developed model for *M/M/R* machine repair problem with spares and additional repairman. Jain et al. (2000) and Shawky (2000) studied a problem with one additional repairman in case of long queue of failed units.

The concept of balking and reneging for machine repair problems in different frameworks was also employed by many researchers working in the field of queueing theory. In recent past, queueing problems with balking and reneging have been studied by Abou E1 Ata (1991), Abou E1 Ata and Hariri (1992) and many others. Jain and Premlata (1994), investigated *M/M/R* machine repair problem with reneging and spares. Shawky (1997) and Jain & Singh (2002) considered machine interference model with balking reneging and additional servers for longer queue. Ke and Wang (1999) developed cost analysis of the *M/M/R* machine repair problem with balking, reneging and server breakdowns. Jain et al. (2003) investigated a queueing model of machining system with balking, reneging, additional repairman and two modes of failure. A1- seedy (2004) presented a queueing model with fixed and variable channel considering balking and reneging concepts. Jain et al. (2005) suggested a loss and delay model for queueing problem with discouregement and additional servers.

In this paper, we study machine repair problem with balking, reneging, spares and additional repairmen by using birth death process. The spares are of two types namely cold and warm. The life times and repair times of the units are assumed to be exponentially distributed. The terminology of the model and notations used are given in section 2. In section 3, the governing equations in steady state and their product type solution are provided. In section 4, some performance measures are derived. In section 5 cost analysis is made. The discussion and ideas for further extension of the work done are given in the last section 6.

2. MODEL DESCRIPTION AND NOTATIONS

Consider mixed multi-components machining system with balking, reneging, spares and additional repainman. For formulating the model mathematically, the following assumptions are made :

* There are M operating, S warm standby and Y cold standby units in the system.
* The system will work with at least m operating units where for normal functioning M units are required.
* The life time and repair time of units are assumed to be exponentially distributed.
* The repair facility consists of C permanent repairmen and r additional removable repairmen to maintain the amount of prodution up to a desired goal. If the number of failed units is more than the permanent repairmen then we employ the additional removable repairmen one by one depending upon work load.
* After repair, the unit will join the standby group. Whin an operating unit fails, it is replaced by cold standby unit if available. If all cold standbys are exhausted, then it is replaced by warm standby unit.
* The repairmen repair the failed units in *FCFS* fashion.
* We assume that $\beta(0 \le \beta \le 1)$ is the probability of the unit to join the queue when all permanent repairmen are busy and some standby units are available and β_0 when all standby are exhausted and no additional repairman turns on. When $j\,(1 \le j \le r)$ additional repairmen are working, the balking probability is given by $1-\beta_j$.
* Failed units renege exponentially with parameter v when all permanent repairmen are busy and standby units are available. In case when all standby unnits are exhausted and number of failed units is below and equal to threshold level T, reneging parameter is denoted by v_0. Unit reneges exponentially with

parameter v_j when all permanent repairmen and $j\,(1 \le j \le r)$ additional repairmen are busy.

* The additional removable repairmen will be available for repair depending upon the number of failed units present in the system according to a prescribed scheme as stated below :
  - When there are $n<T$ failed units, only C permanent repairmen are available for repair them.
  - In case of $jT \le n < (j+1)T$, $j=1, 2,...,r$-1, there are j additional repairmen available to provide repair with rate μ_i. The j^{th} additional repairmen is again removed when queue length drops to jT-1, $j=1,2,...,r$.
  - In case of $rT \le n < M+S+Y-m$ failed units, all $(C+r)$ repairmen i.e. all the permanent and additional repairmen will be busy in the system.

We develop the mathmatical model by taking suitable notations which are given below :

M	:	The number of operating units in the machining system.
C	:	The number of permanent repairmen
r	:	The number of additional removable repairmen
S	:	The number of warm standby units
Y	:	The number of cold standby units
λ	:	Failure rate of operating units
α	:	Failure rate of a warm standby
β	:	Joining probability of a failed units in the queue when some standby units are available.
β_j	:	joining probability of a failed units when all standbys are exhausted and j $(j=0,1...,r)$ additional repairmen are turn on.
v, v_j	:	Reneging parameters of failed units whin a few standbys are aviable, and no standby is avialable and j $(j=0,1...,r)$ additional repairmen are turn on.
μ	:	Repair rate of permanent repairmen.
μ_f	:	Faster repair rate of permanent repairman when all standbys are exhausted.
μ_i	:	Repair rate of i^{th} $(i=1,2,...,r)$ additional removable repairman.
$\lambda(n), \mu(n)$	:	State dependent failure rate, repair rate of units when there are n failed units present in the system.
n	:	The number of failed units in the system waiting for their repair including those failed units which are being repaired.

p_n : Probability that there n failed units present in the system in steady state.

P_0 : Probability that is no failed unit in the system.

3 FORMULATION OF THE PROBLEM. We assume two cases for analysis purpose, which are given as follows :

Case I : $C \le Y$

In this case the failure rates and repair rates of the units are state dependent and are given by

$$\lambda(n)=\begin{cases} M\lambda + S\alpha, & 0 \le n < C \\ M\lambda\beta + S\alpha, & C \le n < Y \\ M\lambda\beta + (S+Y-n)\alpha & Y \le n < Y+S \\ M\lambda\beta_0 + (S+Y-n)\alpha & Y+S \le n < T \\ (M+S+Y-n)\lambda\beta_j, & jT \le n < (j+1)T, j=1,2,\ldots,r \\ (M+S+Y-n)\lambda\beta_r, & rT \le n < M+S+Y-m \end{cases} \quad \ldots(1)$$

and

$$=\mu(n)\begin{cases} n\mu, & 0 < n \le C \\ C\mu + (n-C)\upsilon, & C < n \le Y+S \\ C\mu_f + (n-C)\upsilon_0, & Y+S+1 \le n \le T \\ C\mu_f + \sum_{i=1}^{j}\mu_i + \left(n-\overline{C+j}\right)\upsilon_j, & jT < n \le (j+1)T, j=1,2,\ldots,r-1 \\ C\mu_f + \sum_{i=1}^{r}\mu_i + \left(n-\overline{C+r}\right)\upsilon r, & rT < n \le M+S+Y-m \end{cases}$$

...(2)

Using appropriate state dependent rates given in (1) and (2) we can write the governing steady state equations as :

$$-(M\lambda + S\alpha)p_0 + \mu p_1 = 0 \quad \ldots(3)$$

$$-(M\lambda + S\alpha + n\mu)P_n + (M\lambda + S\alpha)p_{n-1} + (n+1)\mu p_{n+1} = 0, \quad 0<n<C \quad \ldots(4)$$

$$-(M\lambda\beta + S\alpha + C\mu)p_c + (M\lambda + S\alpha)p_{c-1} + (C\mu + V)p_{c+1} = 0 \quad \ldots(5)$$

$$-[M\lambda\beta + S\alpha + C\mu + (n-C)\upsilon]p_n + (M\lambda\beta + S\alpha)p_{n-1} + [C\mu + (n+1-C)\upsilon]P_{n+1} = 0, \quad C < n \le Y \quad \ldots(6)$$

$$-[M\lambda\beta + (S+Y-n)\alpha + C\mu + (n-C)\upsilon]P_n + [M\lambda\beta + (S+Y+1-n)\alpha]P_{n-1}$$
$$+[C\mu + (n+1-C)\upsilon]P_{n+1} = 0, \quad Y \le n < Y+S \quad \ldots(7)$$

$$-\left[M\lambda\beta_0 + C\mu + (Y+S-C)\upsilon\right]P_{Y+S} + (M\lambda\beta + \alpha)P_{Y+S-1}$$

$$+\left[C\mu_f + (Y+S+1-C)\upsilon_0\right]P_{Y+S+1} = 0 \qquad ...(8)$$

$$-\left[M\lambda\beta_0 + (S+Y-n)\alpha + C\mu_f + (n-C)\upsilon_0\right]p_n + \left[M\lambda\beta_0 + (S+Y+1-n)\alpha\right]p_{n-1}$$

$$+\left[C\mu_f + (n+1-C)\upsilon_0\right]p_{n+1} = 0, \qquad Y+S<n<T \qquad ...(9)$$

$$-\left[(M+S+Y-T)\lambda\beta_1 + C\mu_f + (T-C)\upsilon_0\right]P_T + \left[M\lambda\beta_0 + (S+Y+1-T)\alpha\right]P_{r-1}$$

$$+\left[C\mu_f + \mu_1 + (T-C)\upsilon_1\right]P_{T+1} = 0 \qquad ...(10)$$

$$-\left[(M+S+Y-jT)\lambda\beta_j + C\mu_f + \sum_{i=1}^{j-1}\mu_i + \left(jT - \overline{C+j-1}\right)V_{j-1}\right]PjT +$$

$$\left[(M+S+Y+1-jT)\lambda\beta_{j-1}\right]P_{jT-1} + \left[C\mu_f + \sum_{i=1}^{j}\mu_i + \left(jT+1-\overline{C+j}\right)\upsilon_j\right]P_{jT+1} = 0$$

$$j=1,2,\ldots,r-1 \qquad ...(11)$$

$$-\left[(M+S+Y-n)\lambda\beta_j + C\mu_f + \sum_{i=1}^{j}\mu_i + \left(n-\overline{C+j}\right)V_j\right]P_n + \left[(M+S+Y+1-n)\lambda\beta_j\right]P_{n-1} +$$

$$\left[C\mu_f + \sum_{i}^{j}\mu_i + \left(n+1-\overline{C+j}\right)\upsilon_j\right]P_{n+1} = 0 \qquad jT<n<(j+1)T,\ j=1,2,\ldots,r\text{-}1 \qquad ...(12)$$

$$-\left[(M+S+Y-rT)\lambda\beta_r + C\mu_f + \sum_{i=1}^{r-1}\mu_i + \left(rT+1-\overline{C+r}\right)\upsilon_{r-1}\right]P_{rT}$$

$$+\left[(M+S+Y+1-rT)\lambda\beta_{r-1}\right]P_{rT-1} + \left[\mu_f + \sum_{i=1}^{r}\mu_i + \left(rT+1-\overline{C+r}\right)\upsilon_r P_{rT+1} = 0\right] \qquad ...(13)$$

$$-\left[(M+S+Y-n)\lambda\beta_r + C\mu_f + \sum_{i=1}^{r}\mu_i + \left(n-\overline{C+r}\right)\upsilon_r\right]P_n + \left[M+S+Y+1-n)\lambda\beta_r\right]P_{n-1}$$

$$+\left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(n+1-\overline{C+r}\right)\upsilon_r\right]P_{n+1} = 0, \quad rT<n<M+S+Y\text{-}m \qquad ...(14)$$

$$-\left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(M+S+Y-m-\overline{C+r}\right)\upsilon_r\right]P_{m+S+Y-m} + \left[(m+1)\lambda\beta_r\right]P_{M+S+Y-m-1} = 0 \quad ...(15)$$

The steady state solution of equations (3)-(15) by using the product type solution is obtained as follows :

$$
P_n = \begin{cases}
\left(\dfrac{M\lambda+S\alpha}{\mu}\right)^n \dfrac{1}{n!}P_0, & 0<n\le C \\[2ex]
\dfrac{(M\lambda\beta+S\alpha)^{n-c}}{\prod\limits_{k=C+1}^{n}\left[C\mu+(k-C)\upsilon\right]}\left(\dfrac{M\lambda+S\alpha}{\mu}\right)^c \dfrac{1}{C!}P_0, & C<n\le Y \\[2ex]
\dfrac{\prod\limits_{i=Y+1}^{n}\left[M\lambda\beta+(S+Y+1-i)\alpha\right]}{\prod\limits_{k=C+1}^{n}\left[C\mu+(k-C)\upsilon\right]}(M\lambda\beta+S\alpha)^{Y-c}S_1.P_0, & Y<n\le Y+s \\[2ex]
\dfrac{\prod\limits_{i=Y+s+1}^{n}\left[M\lambda\beta_0+(S+Y+1-i)\alpha\right]\prod\limits_{i=Y+1}^{Y+S}\left[M\lambda\beta+(S+Y+1-i)\alpha\right]}{\prod\limits_{k=Y+S+1}^{n}\left[C\mu_f+(k-C)\upsilon_0\right]\prod\limits_{k=C+1}^{Y+S}\left[C\mu+(k-C)\upsilon\right]}S_2.P_0, & Y+S<n\le T \\[2ex]
\dfrac{\prod\limits_{i=T+1}^{n}(M+S+Y+1-i)(\lambda)^{n-jT}(\beta_j)^{n-jT}}{\prod\limits_{k=c+1}^{n-jT}[C\mu_f+\sum\limits_{i=1}^{j}\mu_i+(K-\overline{C+j})\upsilon_j]\left[\prod\limits_{1=1}^{j-1}\prod\limits_{k=1T+1}^{(1+1)T}\left[C\mu_f+\sum\limits_{i=1}^{j}\mu_i+(K-\overline{C+1})\upsilon_1\right]\right]} & \\
\quad\times\dfrac{\prod\limits_{i=Y+S+1}^{T}\left[M\lambda\beta_0+(S+Y+1-i)\alpha\right]\left(\prod\limits_{i=1}^{j-1}(\lambda\beta_i)^T\right)}{\prod\limits_{k=Y+S+1}^{T}\left[C\mu_f+(k-C)\upsilon_0\right]}S_3P_0, & jT<n\le(j+1)T, 1\le j<r \\[2ex]
\dfrac{\prod\limits_{i=T+1}^{n}(M+S+Y+1-i)(\lambda\beta_r)^{n-rT}\left(\prod\limits_{i=1}^{r-1}(\lambda\beta_i)^T\right)}{\prod\limits_{k=rT+1}^{n}[C\mu_f+\sum\limits_{i=1}^{r}\mu_i+(K-\overline{C+r})\upsilon_r]\prod\limits_{k=jT+1}^{rT}\left[C\mu_f+\sum\limits_{i=1}^{r}\mu_i+(K-\overline{C+r})\upsilon_r\right]} & \\
\quad\times\dfrac{1}{\left[\prod\limits_{1=1}^{r-1}\prod\limits_{k=lT+1}^{(1+1)T}\left(C\mu_f+\sum\limits_{i=1}^{1}\mu_i+(K-\overline{C+1})\upsilon_1\right)\right]}S_4.P_0, & rT<n\le M+S+Y-m
\end{cases}
$$

...(16)

where

$$S_1=\left(\frac{M\lambda+S\alpha}{\mu}\right)^C\frac{1}{C!}, \qquad S_2=(M\lambda\beta+S\alpha)^{Y-C}S_1$$

$$S_3 = \frac{\prod_{i=Y+1}^{Y+S}[M\lambda\beta + (S+Y+1-i)\alpha]}{\prod_{k=C+1}^{Y+S}[C\mu + (k-C)\upsilon]} S_2, \qquad S_4 = \frac{\prod_{i=Y+S+1}^{T}[M\lambda\beta_0 + (S+Y+1-i)\alpha]}{\prod_{k=Y+S+1}^{T}[C\mu_r + (k-C)\upsilon_0]} S_3 .$$

Now we determine P_0, using the normalization condition $\sum_{n=0}^{M+S+Y-m} P_n = 1$. Now we get

$$P_0^{-1} = \sum_{n=0}^{C} \frac{(M\lambda + S\alpha)^n}{\mu^n} \frac{1}{n!} + \frac{(M\lambda + S\alpha)^C}{\mu^C} \frac{1}{C!} \sum_{K=C+1}^{Y} \frac{(M\lambda\beta + S\alpha)^{n-C}}{\prod_{k=C+1}^{n}[C\mu + (k-C)\upsilon]}$$

$$+ S_1 (M\lambda + S\alpha)^{Y-C} \sum_{n=Y+1}^{Y+S} \frac{\prod_{i=Y+1}^{n}[M\lambda\beta + (S+Y+1-i)\alpha]}{\prod_{k=C+1}^{n}[C\mu_r + (k-C)\upsilon]} + S_2 \frac{\prod_{i=Y+1}^{Y+S}[M\lambda\beta + (S+Y+1-i)\alpha]}{\prod_{k=C+1}^{Y+S}[C\mu + (k-C)\upsilon]}$$

$$\sum_{n=Y+S+1}^{T} \frac{\prod_{i=Y+S+1}^{n}[M\lambda\beta_0 + (S+Y+1-i)\alpha]}{\prod_{k=Y+S+1}^{n}[C\mu_r + (k-C)\upsilon_0]} + S_3 \frac{\prod_{i=Y+S+1}^{T}[M\lambda\beta_0 + (S+Y+1-i)\alpha]\left(\prod_{i=1}^{j-1}(\lambda\beta_i)^T\right)}{\prod_{k=Y+S+1}^{T}[C\mu_f + (k-C)\upsilon_0]}$$

$$\frac{1}{\left[\prod_{l=1}^{j-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f + \sum_{i=1}^{l}\mu_i + \left(k - \overline{C+1}\right)\upsilon_1\right)\right]} \sum_{j=1}^{r-1}\sum_{n=jT+1}^{(j+1)T} \frac{\prod_{i=T+1}^{n}(M+S+Y+1-i)(\lambda)^{n-jT}\left(\beta_j\right)^{n-jT}}{\prod_{k=jT+1}^{n}\left[C\mu_r + \sum_{i=1}^{i}\mu_i + \left(k - \overline{C+j}\right)\upsilon_j\right]}$$

$$+ S_4 \frac{\left(\prod_{i=1}^{r-1}(\lambda\beta_i)^T\right)}{\prod_{k=jT+1}^{rT}\left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(k - \overline{C+r}\right)\upsilon_r\right]} \frac{1}{\left[\prod_{l=1}^{r-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f + \sum_{i=1}^{l}\mu_i + \left(k - \overline{C+1}\right)\upsilon_l\right)\right]}$$

$$\sum_{n=rT+1}^{M+S+Y-m} \frac{\prod_{i=T+1}^{n}(M+S+Y+1-i)(\lambda\beta_r)^{n-rT}}{\prod_{k=rT+1}^{n}\left[C\mu_f+\sum_{i=1}^{r-1}\mu_i+\left(k-\overline{C+r}\right)\upsilon_r\right]} \qquad \text{...(17)}$$

Case-II : C>Y

In this case the failure rate and repair rate are as follows:

$$\lambda(n)=\begin{cases} M\lambda+S\alpha, & 0\le n<Y \\ M\lambda+(S+Y-n)\alpha & Y\le n<C \\ M\lambda\beta+(S+Y-n)\alpha, & C\le n<Y+S \\ M\lambda\beta_0+(S+Y-n)\alpha, & Y+S\le n<T \\ (M+S+Y-n)\lambda\beta_j, & jT\le n<(j+1)T,\ j=1,2,\ldots,r \\ (M+S+Y-n)\lambda\beta_r, & rT\le n<M+S+Y-m \end{cases} \qquad \text{... (18)}$$

and

$$\mu(n)=\begin{cases} n\mu & 0<n\le C \\ C\mu+(n-C)\upsilon, & C<n\le Y+S \\ C\mu_r+(n-C)\upsilon_0, & Y+S+1\le n\le T \\ C\mu_f+\sum_{i=1}^{j}\mu_i+\left(n-\overline{C+j}\right)\upsilon_j & jT<n\le(j+1)T,\ j=1,2,\ldots,r-1 \\ C\mu_f+\sum_{i=1}^{r}\mu_i+\left(n-\overline{C+r}\right)\upsilon_r, & rT<n\le M+S+Y-m \end{cases} \qquad \text{...(19)}$$

Using (18) and (19) we can write the governing steady state equations as :

$$-(M\lambda+S\alpha)P_0+\mu P_1=0 \qquad \text{...(20)}$$

$$-[M\lambda+S\alpha+n\mu]P_n+(M\lambda+S\alpha)P_{n-1}+(n+1)\mu P_{n+1}=0, \qquad 0<n\le Y \qquad \text{...(21)}$$

$$-[M\lambda+(S+Y-n)\alpha+n\mu]P_n+[M\lambda+(S+Y+1-n)\alpha]P_{n-1}+(n+1)P_{n+1}=0, \qquad \text{...(22)}$$

$$-[M\lambda\beta+(S+Y-C)\alpha+C\mu]P_C+[M\lambda+(S+Y+1-C)\alpha]P_{C-1}+(C\mu+\upsilon)P_{C+1}=0, \qquad \text{...(23)}$$

$$-[M\lambda\beta+(S+Y-n)\alpha+C\mu+(n-C)\upsilon]P_n+[M\lambda\beta+(S+Y+1-n)\alpha]P_{n-1}$$

$$+[C\mu+(n+1-C)\upsilon]P_{n+1}=0, \qquad C<n<Y+S \qquad \text{...(24)}$$

$$-[M\lambda\beta_0+C\mu+(Y+S-C)\upsilon]P_{Y+S}+(M\lambda\beta+\alpha)P_{Y+S-1}+\left[C\mu_f+(Y+S+1-C)\upsilon_0\right]P_{Y+S+1}=0 \qquad \text{...(25)}$$

$$-\left[M\lambda\beta_0 + (S+Y-n)\alpha + C\mu_f + (n-C)\upsilon_0\right]P_n + \left[M\lambda\beta_0 + (S+Y+1-n)\alpha\right]P_{n-1}$$

$$+\left[C\mu_f + (n+1-C)\upsilon_0\right]P_{n+1} = 0, \qquad Y+S<n<T \qquad \text{...(26)}$$

$$-\left[(M+S+Y-T)\lambda\beta_1 + C\mu_f + (T-C)\upsilon_0\right]P_T + \left[M\lambda\beta_0 + (S+Y+1-T)\alpha\right]P_{T-1}$$

$$+\left[C\mu_f + \mu_1 + (T-C)\upsilon_1\right]P_{T+1} = 0 \qquad \text{...(27)}$$

$$-\left[(M+S+Y-jT)\lambda\beta_j + C\mu_f\sum_{i=1}^{j-1}\mu_i + \left(jT-\overline{C+j-1}\right)\upsilon_{j-1}\right]P_{jT} + \left[(M+S+Y+1-jT)\lambda\beta_{j-1}\right]$$

$$P_{jT-1} + \left[C\mu_f + \sum_{i=1}^{j}\mu_i + \left(jT+1-\overline{C+j}\right)\upsilon_j\right]P_{jT+1} = 0 \quad j=1,2,\ldots,r-1 \qquad \text{...(28)}$$

$$-\left[(M+S+Y-n)\lambda\beta_j + C\mu_f + \sum_{i=1}^{j}\mu_i + \left(n-\overline{C+j}\right)\upsilon_j\right]P_n + \left[(M+S+Y+1-n)\lambda\beta_j\right]P_{n-1}$$

$$+\left[C\mu_f + \sum_{i=1}^{j}\mu_i + \left(n+1-\overline{C+j}\right)\upsilon_j\right]P_{n+1} = 0, \qquad jT<n<(j+1)T j=1,2,\ldots,r-1 \qquad \text{...(29)}$$

$$-\left[(M+S+Y-rT)\lambda\beta_r + C\mu_f + \sum_{i=1}^{r-i}\mu_i + \left(rT+1-\overline{C+r}\right)\upsilon_{r-1}\right]P_{rT}$$

$$+\left[(M+S+Y+1-rT)\lambda\beta_{r-1}\right]P_{rT-1} + \left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(rT+1-\overline{C+r}\right)\upsilon_r\right]P_{rT+1} = 0 \qquad \text{...(30)}$$

$$-\left[(M+S+Y-n)\lambda\beta_r + C\mu_f + \sum_{i=1}^{r}\mu_i + \left(n-\overline{C+r}\right)\upsilon_r\right]P_n + \left[(M+S+Y+1-n)\lambda\beta_r\right]P_{n-1}$$

$$+\left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(n+1-\overline{C+r}\right)\upsilon_r\right]P_{n+1} = 0, \qquad rT<n<M+S-Y-m \qquad \text{...(31)}$$

$$-\left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(M+S+Y-m-\overline{C+r}\right)\upsilon_r\right]P_{M+S+Y-m} + \left[(m+1)\lambda\beta_r\right]P_{M+S+Y-m-1} = 0 \quad \text{...(32)}$$

The steady state solution of above equations by using the product type solution is obtained as follows :

$$
P_n = \begin{cases}
\dfrac{(M\lambda + S\alpha)^n}{\mu^n}\dfrac{1}{n!}P_0, & 0 < n \le Y \\[2ex]
\dfrac{\prod\limits_{i=Y+1}^{n}\left[M\lambda + (S+Y+1-i)\alpha\right]}{\left(\prod\limits_{k=Y+1}^{n} k\right)\mu^n}\dfrac{\left[M\lambda + S\alpha\right]^Y}{Y!}P_0, & Y < n \le C \\[2ex]
\dfrac{\prod\limits_{i+C+1}^{n}\left[M\lambda\beta + (S+Y+1-i)\alpha\right]}{\prod\limits_{k=C+1}^{n}\left(C\mu + (k-C)\upsilon\right)}\dfrac{\prod\limits_{i=Y+1}^{C}\left[M\lambda + (S+Y+1-i)\alpha\right]}{\left(\prod\limits_{k=Y+1}^{C} k\right)\mu^C}S_5P_0, & C < n \le Y+S \\[2ex]
\dfrac{\prod\limits_{i=Y+S+1}^{n}\left[M\lambda\beta_0 + (S+Y+1-i)\alpha\right]}{\prod\limits_{k=Y+S+1}^{n}\left[C\mu_f + (k-C)\upsilon_0\right]}\dfrac{\prod\limits_{i=C+1}^{Y+S}\left[M\lambda\beta + (S+Y+1-i)\alpha\right]}{\prod\limits_{k=C+1}^{Y+S}\left(C\mu + (k-C)\upsilon\right)}S_6P_0, & Y+S < n \le T \\[2ex]
\dfrac{\prod\limits_{i=T+1}^{n}(M+S+Y+1-i)(\lambda)^{n-jT}\left(\prod\limits_{i=1}^{j-1}(\lambda\beta_i)^T\right)(\beta_j)^{n-jT}}{\prod\limits_{k=jT+1}^{n}\left[C\mu_f + \sum\limits_{i=1}^{j}\mu_i + \left(k-\overline{C+1}\right)\upsilon_j\right]}\dfrac{\prod\limits_{i=Y+S+1}^{T}\left[M\lambda\beta_0 + (S+Y+1-i)\alpha\right]}{\prod\limits_{k=Y+S+1}^{T}\left(C\mu_f + (k-C)\upsilon_0\right)} & \\
\quad\times\dfrac{1}{\left[\prod\limits_{l=1}^{j-1}\prod\limits_{k=lT+1}^{(l+1)T}\left(C\mu_f + \sum\limits_{i=1}^{l}\mu_i + \left(k-\overline{C+1}\right)\upsilon_l\right)\right]}S_7P_0, & jT < n \le (j+1)T, 1 \le j < r \\[2ex]
\dfrac{\prod\limits_{i=T+1}^{n}(M+S+Y-i)\left(\prod\limits_{i=1}^{r-1}(\lambda\beta_i)^T\right)(\lambda\beta_r)^{n-rT}}{\prod\limits_{k=rT+1}^{n}\left[C\mu_f + \sum\limits_{i=1}^{r}\mu_i + \left(k-\overline{C+r}\right)\upsilon_r\right]}\dfrac{1}{\prod\limits_{k=jT+1}^{rT}\left[C\mu_f + \sum\limits_{i=1}^{r}\mu_i + \left(k-\overline{C+r}\right)\upsilon_r\right]} & \\
\quad\times\dfrac{1}{\left[\prod\limits_{l=1}^{r-1}\prod\limits_{k=lT+1}^{(l+1)T}\left(C\mu_f + \sum\limits_{i=1}^{l}\mu_i + \left(k-\overline{C+1}\right)\upsilon_1\right)\right]}S_8P_0, & rT < n \le M+S+Y-m
\end{cases}
$$

...(33)

where

$$
S_5 = \frac{(M\lambda + S\alpha)^Y}{Y!}, \qquad S_6 = \frac{\prod\limits_{i=Y+1}^{C}\left[M\lambda + (S+Y+1-i)\alpha\right]}{\left(\prod\limits_{k=Y+1}^{C}(k)\right)\mu^C}S_5
$$

$$S_7 = \frac{\prod_{i=C+1}^{Y+S}\left[M\lambda\beta + (S+Y+1-i)\alpha\right]}{\prod_{k=C+1}^{Y+S}\left(C\mu + (k-C)\upsilon\right)} S_6, \quad S_8 = \frac{\prod_{i=Y+S+1}^{T}\left[M\lambda\beta_0 + (S+Y+1-i)\alpha\right]}{\prod_{k=Y+S+1}^{T}\left[C\mu_f + (k-C)\upsilon_0\right]} S_7 .$$

Now we determine P_0, using the normalization condition $\sum_{n=0}^{M+S+Y-m} P_n = 1$. Then, we get

$$P_0^{-1} = \sum_{n=0}^{Y} \frac{(M\lambda + S\alpha)^n}{\mu^n}\frac{1}{n!} + \frac{(M\lambda + S\alpha)^Y}{Y!} \sum_{n=Y+1}^{C} \frac{\prod_{i=Y+1}^{n}\left[M\lambda + (S+Y+1-i)\right]\alpha}{\left(\prod_{k=Y+1}^{n}(k)\right)\mu^n}$$

$$+ S_5 \frac{\prod_{i=Y+1}^{C}\left[M\lambda + (S+Y+1-i)\alpha\right]}{\left(\prod_{k=Y+1}^{C} k\right)\mu^C} \sum_{n=C+1}^{Y+S} \frac{\prod_{i=C+1}^{n}\left[M\lambda\beta + (S+Y+1-i)\alpha\right]}{\prod_{k=C+1}^{n}\left(C\mu + (k-C)\upsilon\right)}$$

$$+ S_6 \prod_{i=C+1}^{Y+S} \frac{\left[M\lambda\beta + (S+Y+1-i)\alpha\right]}{\prod_{k=C+1}^{Y+S}\left(C\mu + (k-C)\upsilon\right)} \sum_{n=Y+S+1}^{T} \frac{\prod_{i=Y+S+1}^{n}\left[M\lambda\beta_0 + (S+Y+1-i)\alpha\right]}{\prod_{k=Y+S+1}^{n}\left[C\mu_f + (k-C)\upsilon_0\right]}$$

$$+ S_7 \frac{\prod_{i=Y+S+1}^{T}\left[M\lambda\beta_0 + (S+Y+1-i)\alpha\right]\left(\prod_{i=1}^{j-1}(\lambda\beta_i)^T\right)}{\prod_{k=Y+S+1}^{T}\left[C\mu_f + (k-C)\upsilon_0\right]} \frac{1}{\left[\prod_{l=1}^{j-1}\prod_{k=1T+1}^{(l+1)T}\left(C\mu_f + \sum_{i=1}^{l}\mu_i + \left(k - \overline{C+1}\right)\upsilon_1\right)\right]}$$

$$\times \sum_{j=1}^{r-1}\sum_{n=jT+1}^{(j+1)T} \frac{\prod_{i=T+1}^{n}(M+S+Y+1-i)(\lambda)^{n-jT}\left(\beta_j\right)^{n-jT}}{\prod_{k=jT+1}^{n}\left[C\mu_f + \sum_{i=1}^{j}\mu_i + \left(k - \overline{c+j}\right)\upsilon_j\right]} + S_8 \frac{\left(\prod_{i=1}^{r-1}(\lambda\beta_i)^T\right)}{\prod_{k=jT+1}^{rT}\left[C\mu_f + \sum_{i=1}^{r}\mu_i + \left(k - \overline{C+r}\right)\upsilon_r\right]}$$

$$\times \frac{1}{\left[\prod_{l=1}^{r-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f+\sum_{i=1}^{l}\mu_i+\left(k-\overline{C+1}\right)\right)\upsilon_1\right]}\sum_{n=rT+1}^{M+S+Y-m}\frac{\prod_{i=T+1}^{n}(M+S+Y+1-i)(\lambda\beta_r)^{n-rT}}{\prod_{k=rT+1}^{n}\left[C\mu_f+\sum_{i=1}^{r-1}\mu_i+\left(k-\overline{C+r}\right)\upsilon_r\right]} \quad (34)$$

4. Some Performance Measures. After obtaining queue size distribution in previous section, now we obtain some system characteristics as follows :

* The expected number of failed units in the system

$$E(n)=\sum_{n=1}^{M+S+Y-m} n.P_n\,. \qquad ...(35)$$

* Expected number of operating units in the system

$$E(O)=M-\sum_{n=Y+S+1}^{M+Y+S-m}(n-Y+S)P_n\,. \qquad ...(36)$$

* Expected number of unused cold spare units in the system

$$E(UCS)=\sum_{n=0}^{Y}(Y-n)P_n\,. \qquad ...(37)$$

* Expected number of unused warm spare units in the system

$$E(UYS)=Y\sum_{n=0}^{S}P_n+\sum_{n=Y+1}^{Y+S}(Y+S-n)P_n\,. \qquad ...(38)$$

* Expected number of idle permanent repairmen

$$E(I)=\sum_{n=0}^{C-1}(C-n)P_n\,. \qquad ...(39)$$

* Expected number of busy permanent repairmen

$$E(B)=C-E(I)\,. \qquad ...(40)$$

* Probability of j^{th} $(1\le j\le r-1)$ additional repairman being busy

$$E(A_j)=\sum_{j=1}^{r-1}j\sum_{n=jT+1}^{(j+1)T}P_n\,. \qquad ...(41)$$

* Expected number of busy additional removable repairmen

$$E(BAS)=\sum_{j=1}^{r-1}j\sum_{n=jT+1}^{(j+1)T}P_n+r\sum_{n=rT+1}^{M+S+Y-m}P_n\,. \qquad ...(42)$$

5. Special Cases

Case I : Model with Cold Spares, Balking Reneging and Additional Repairmen. If $S=0$ then our model reduces to model with cold spares, balking, reneging and having additional repairmen.

(a) For the case $C \le Y$,

we determine steady state probability distribution as

$$P_n = \begin{cases} \dfrac{(M\rho)^n}{n!} P_0, & 0 < n \le C \\ \dfrac{(M\lambda\beta)^{n-C}}{\prod_{k=C+1}^{n}[C\mu + (k-C)\upsilon]} A.P_0, & C < n \le Y \\ \dfrac{(M\lambda\beta_0)^{n-Y}}{\prod_{k=Y+1}^{n}[C\mu_f + (k-C)\upsilon_0]} B.P_0, & Y < n \le T \\ \dfrac{\prod_{i=T+1}^{n}(M+Y+1-i)\left(\prod_{i=1}^{j-1}(\lambda\beta_i)^T\right)(\lambda\beta_j)^{n-jT}}{\prod_{k=jT+1}^{n}\left[C\mu_f + \sum_{i=1}^{j}\mu_i + (k-\overline{C+j})\upsilon_j\right]} \dfrac{1}{\left[\prod_{1=1}^{j-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f + \sum_{i=1}^{l}\mu_i + (k-\overline{C+1})\upsilon_1\right)\right]} C.P_0, & jT < n \le (j+1)T, 1 \le j < r \\ \dfrac{\prod_{i=T+1}^{r}(M+Y+1-i)\left(\prod_{i=1}^{r-1}(\lambda\beta_i)^T\right)(\lambda\beta_r)^{n-rT}}{\prod_{k=rT+1}^{n}\left[C\mu_f + \sum_{i=1}^{r-1}\mu_i + (k-\overline{C+r})\upsilon_r\right]} \dfrac{1}{\left[\prod_{1=1}^{r-1}\prod_{k=lT+1}^{(l+1)T}\left(C\mu_f + \sum_{i=1}^{l}\mu_i + (k-\overline{C+1})\upsilon_1\right)\right]} DP_0, & rT < n \le M+Y-m \end{cases}$$

where

$$\rho = \frac{\lambda}{\mu}, \qquad A = \frac{(M\rho)^C}{C!}, \qquad B = \frac{(M\lambda\beta)^{Y-C}}{\prod_{k=C+1}^{Y}[C\mu + (k-C)\upsilon]} A,$$

$$C = \frac{(M\lambda\beta_0)^{T-Y}}{\prod_{k=Y+1}^{T}[C\mu_f + (k-C)\upsilon_0]} B, \qquad D = \frac{C}{\prod_{k=jT+1}^{rT}\left[C\mu_f + \sum_{i=1}^{r}\mu_i + (k-\overline{C+r})\upsilon_r\right]} \qquad \text{...(43)}$$

Case-II : Model with Balking, Reneging and Mixed Spares. By setting $r=0$ our model reduces to machining system with mixed spares, balking and reneging.

Now for case $C>Y$, we get steady state probability distribution as

$$P_n = \begin{cases} \left(\dfrac{M\lambda + S\alpha}{\mu}\right)^n \dfrac{1}{n!} P_0, & 0 < n \le Y \\ \dfrac{(M\lambda + S\alpha)^Y}{Y!} \dfrac{\prod_{i=Y+1}^{n}[M\lambda + (S+Y+1-i)\alpha]}{\left(\prod_{k=Y+1}^{n}(k)\right)} \dfrac{1}{\mu^n} P_0, & Y < n \le C \\ \dfrac{\prod_{i=Y+1}^{C}[M\lambda + (S+Y+1-i)\alpha]x}{\left(\prod_{k=Y+1}^{C} k\right)\mu^C} \dfrac{\prod_{k=C+1}^{n}[M\lambda\beta + (S+Y+1-i)\alpha]}{\left(\prod_{k=C+1}^{n}(C\mu + (k-C)\upsilon)\right)} S_5 P_0, & C < n \le Y+S \\ \dfrac{\prod_{i=C+1}^{Y+S}\left[M\lambda\beta + \left(S+Y+1-\overline{C+i}\right)\alpha\right]}{\left(\prod_{k=C+1}^{Y+S}(C\mu + (k-C)\upsilon)\right)} \dfrac{\prod_{i=Y+S+1}^{n}[M\lambda\beta_0 + (S+Y+1-i)\alpha]}{\prod_{k=Y+S+1}^{n}\left[C\mu_f + (k-C)\upsilon_0\right]} S_6.P_0, & Y+S < n \le T \end{cases} \quad ...(44)$$

Case III. If $\beta = \upsilon = 0$ then our model reduces to Moses (2005) model for machine repair problem with mixed spares and additional repairman.

Case IV. For $Y=0$, $S=0$, $\beta=0$ $\upsilon=0$, we get results for classical machine repair problem discussed by Kleinrock (1985).

6. Cost Function Our main aim in this section is to provide a cost function, which can be minimized to determine the optimal number of repairmen and spares. The average total cost is given by

$$E(C) = C_M \sum_{n=0}^{Y+S} MP_n + C_1 E(I) + C_{SC} E(UCS) + C_{SW} E(UYS) + C_B E(B) + \sum_{j=1}^{r} CA_j E(A_j) \quad (45)$$

where

C_M = Cost per unit time of an operating unit when system works is normal mode.

C_1 = Cost per unit time per idle permanent repairmen.

C_{SC} = Cost per unit time for providing a cold spare unit

C_{Sw} = Cost for unit time for providing a warm spare unit

C_B = Cost per unit time per permanent repairman when he is busy in providing repair.

CA_j = Cost per unit time of $j^{th} (j=1,2,...,r)$ additional repairman.

7. Discussion. In this study, we have developed machine repair model with balking reneging, spares and additional repairmen. The machining system considered consists of warm and cold standby spares along with a repair facility having both permanent and additional repairmen. The provision of spares and additional repairmen may help the system organizer in providing regular magnitude of production up to a desired grade of demand in particular when number of failed units is large. The expressions for several system characteristics and cost function are derived explicitly which can be further employed to find out the optimal combination of spares and repairmen and might be helpful for system designer to determine appropriate system descriptors at optimum cost subject to availability constriant.

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A NOTE ON *MHD* RAYLEIGH FLOW OF A FLUID OF EQUAL KINEMATIC VISCOSITY AND MAGNETIC VISCOSITY PAST A PERFECTLY CONDUCTING POROUS PLATE

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ABSTRACT

The paper deals with the flow of a viscous incompressible fluid of small electrically conductivity past an infinite porous plate started impulsively from rest in presence of a constant transverse magnetic field in fixed relation to the fluid with the imposition of small uniform suction or injection velocity at the plate. Suction or injection velocity at the plate has been calculated using the Laplace transform method. The *MHD* unidirectional flow of a viscous incompressible fluid of small electrical conductivity near an infinite flat plate started impulsively from the rest, which was first studied by Lord Rayleigh [8], has been shown to be self-superposable and an irrotational flow on which it is superposable is determined. Some observations have been made about the vorticity and stream functions of the flow by using the properties of superposability and self-superposability. Vorticity profiles have been plotted and studied for different conditions and for suction and injection.

2000 Mathematics Subject Classification : Primary 76A10; Secondary 76B47.

Keywords : *MHD* Rayleigh flow, KinematicViscosity, Magnetic Viscocity

1. Introduction. The folw about an infinite flat plate which executes linear harmonic oscillation parallel to itself was studied by Stokes [11] and Rayleigh [8]. The impulsive motion of an inffinite flat plate in a viscous incompressible magnetic fluid in the presence of an external magnetic field was studied by. Rossow [9]. Nath [14] studied the Rayleigh problem in slip flow with transverse magnetic field. In the present paper we have obtained the exact solution for the *MHD* flow of a fluid of equal kinematic viscosity and magnetic viscocity past a perfectly conducting porous plate by using the Laplace transform technique. Vorticity and self superposability of the above *MHD* Rayleigh flow have also been studied.

2. Formulation of the problem. Let us consider the unsteady motion of a semi-infinite mass of incompressible, viscous, perfectly conducting fluid past an infinite plate. Let x-axis be along the plate parallel to the flow direction, y-axis perpendicular to the plate and z-axis perpendicular to both the x-axis and y-axis. The imposed magnetic field $\overline{H}_0$ is applied in the direction of y-axis. Let V_s represents the suction velocity at the plate, then by equation of the continuity

$$\frac{\partial v}{\partial y} = 0 .$$

Also the condition, that at $y=0$, $v=v_s$ leads to every where. Due to motion a magnetic field H_x is introduced in the flow direction and from the symmetry of the problem all physical variables will be functions of y and time t only. Let the plate be started impulsively from rest with a constant velocity U, and subject to the conditions :

at $\quad t=0, \quad u=H_x=0, y>0$

at $\quad y=0, \quad u=U, \quad H_x=0, t>0$

as $\quad y\to\infty, u=0, \quad H_x=0$ . ...(2.1)

The differential equations governing the fluid motion are given by [13]

$$\frac{\partial u_1}{\partial t} + v_s \frac{\partial u_1}{\partial y} = \alpha \frac{\partial u_1}{\partial y} = v \frac{\partial^2 u_1}{\partial y^2} \qquad ...(2.2)$$

$$\frac{\partial u_2}{\partial t} + v_s \frac{\partial u_2}{\partial y} = -\alpha \frac{\partial u_2}{\partial y} + v \frac{\partial^2 u_2}{\partial y^2} \qquad ...(2.3)$$

where

$$u_1 = u + \aleph H_x \qquad ...(2.4)$$

$$u_2 = u - \aleph H_x \qquad ...(2.5)$$

u is the x-component of fluid velocity.

λ = Magnetic viscosity of the fluid, $\alpha = \sqrt{\mu H_0 / p}$

η = Coefficient of the viscosity of the fluid, $\aleph = \sqrt{\mu/\rho}$

μ = Permeability of the medium.

Applying Laplace Transform to equations (2.2) and (2.3) we get,

$$\frac{d^2\overline{u}_1}{dy^2} + \frac{(\alpha - v_s)}{v}\frac{d\overline{u}_1}{dy} = \frac{p\overline{u}_1}{v} \qquad ...(2.6)$$

and

$$\frac{d^2\bar{u}_2}{dy^2}-\frac{(\alpha-v_s)}{v}\frac{d\bar{u}_2}{dy}=\frac{p\bar{u}_2}{v}, \qquad ...(2.7)$$

where $\bar{u}_1, \bar{u}_2$ are the Laplace transform of u_1 and u_2 respectively and p is the kernel of the Laplace transform. The solution of equation (2.6) and (2.7) are given by

$$\bar{u}_1 = A\exp\left\{-\frac{(\alpha-v_s)y}{2v}+\frac{y}{2}\sqrt{\left(\frac{\alpha-v_s}{v}\right)^2+4\frac{p}{v}}\right\}+B\exp\left\{-\frac{(\alpha-v_s)y}{2v}-\frac{y}{2}\sqrt{\left(\frac{\alpha-v_s}{v}\right)^2+4\frac{p}{v}}\right\}$$

or

$$\bar{u}_1 = \exp\left(-\frac{(\alpha-v_s)y}{2v}\right)\left\{A\exp\left(\frac{y}{2}\sqrt{\left(\frac{(\alpha-v_s)}{v}\right)^2+4\frac{p}{v}}\right)+B\exp\left\{-\frac{y}{2}\sqrt{\left(\frac{\alpha-v_s}{v}\right)^2+4\frac{p}{v}}\right\}\right\} \quad (2.8)$$

and

$$\bar{u}_2 = \exp\left(-\frac{(\alpha-v_s)y}{2v}\right)\left\{C\exp\left(\frac{y}{2}\sqrt{\left(\frac{(\alpha+v_s)}{v}\right)^2+4\frac{p}{v}}\right)+D\exp\left\{-\frac{y}{2}\sqrt{\left(\frac{\alpha+v_s}{v}\right)^2+4\frac{p}{v}}\right\}\right\} \quad (2.9)$$

Applying Laplace Transform to initial conditions we get,

at $\quad y=0, t>0 \quad \bar{u}_1=\frac{U}{P}, \bar{u}_2=\frac{U}{P}$

and as $\quad y\to\infty, \bar{u}=0, \quad H_x=0.$

We have

$A=C=0$ and $B=D=U.$

Equation (2.2) and (2.3) then become

$$\bar{u}_1=\frac{U}{p}\exp\left\{-\frac{(\alpha-v_s)y}{2v}-\frac{y}{2}\sqrt{\left(\frac{(\alpha-v_s)}{v}\right)^2+4\frac{p}{v}}\right\} \qquad ...(2.10)$$

$$\bar{u}_2=\frac{U}{p}\exp\left\{\frac{(\alpha+v_s)y}{2v}-\frac{y}{2}\sqrt{\left(\frac{(\alpha-v_s)}{v}\right)^2+4\frac{p}{v}}\right\} \qquad ... (2.11)$$

Taking inverse Laplace Transform [7] of equations (2.10) and (2.11) and

then substituting the values of u_1 and u_2 in (2.4) and (2.5) finally we get

$$u=\frac{U}{2}\left\{erfc\left(\frac{y+(\alpha-\upsilon_s)t}{2\sqrt{\upsilon t}}\right)+erfc\left(\frac{y-(\alpha+\upsilon_s)t}{2\sqrt{\upsilon t}}\right)\right\} \qquad \text{...(2.12)}$$

and

$$H_x=\frac{U}{2}\sqrt{\frac{\rho}{\mu}}\left\{erfc\left(\frac{y+(\alpha-\upsilon_s)t}{2\sqrt{\upsilon t}}\right)-erfc\left(\frac{y-(\alpha+\upsilon_s)t}{2\sqrt{\upsilon t}}\right)\right\} \qquad \text{...(2.13)}$$

flow equation indicates the absence of exponential factor which means that there is no Hartmann layer in the ultimate state.

3. Flow Superpsoabile on Rayleigh Flow. Let us suppose that a flow

$$\bar{\upsilon}=(\upsilon_x,\upsilon_y,\upsilon_z) \qquad \text{...(3.1)}$$

is superposable on the flow (2.12). Here υ_x, υ_y, υ_z are independent of x and z i.e., these are either functions of y alone or constant.

Applying the conditions of superposability of two flows, laid down by Ballabh [2] i.e., the two flows with velocity $\bar{\upsilon}_1$ and $\bar{\upsilon}_2$ are mutually superposable to each other if

$$curl[\bar{\upsilon}_1\times curl\bar{\upsilon}_2+\bar{\upsilon}_2\times curl\bar{\upsilon}_1]=0 \qquad \text{...(3.2)}$$

we get,

$$\upsilon_y=\frac{A}{(\partial u/\partial y)}, \qquad \text{...(3.3)}$$

where A is constant.

If $\alpha=0$, $v_s=0$ i.e., when there is no suction or jnjection and magnetic field, we get

$$\upsilon_y=-A\sqrt{\pi\upsilon t}\exp(y^2/4\upsilon t) \qquad \text{...(3.4)}$$

If we consider the motion in z–x plane only and υ_z= constant, then from (3.1) we have

$$\upsilon=\left(0,-A\sqrt{\pi\upsilon t}\exp(y^2/4\upsilon t),\upsilon_z\right) \qquad \text{...(3.5)}$$

From (3.5), we readily have

$Curl\,\bar{\upsilon}=0.$

This means that the motion denoted by (3.5) is irrotational. Hence we can say that an irrotational flow denoted by (3.5) is superposable on the flow (2.12) under the condition $\alpha=0,\upsilon_s=0$.

It was shown by Ballabh [3] that an irrotational flow is superposable on a

rotational one if and only if the vorticity of the latter is constant along the stream lines of the former.

The equation of the stream lines of the motion (1.12) can be deduced as

$x =$ *constant* and $z = Derf(y)$ . ...(3.6)

Hence the vorticity of the Rayleigh flow is constant along the curve (3.6).

4. Self Superposability of the Flow. from equation (2.12) we have

$$curl[\bar{u} \times curl\bar{u}] = 0. \qquad ...(4.1)$$

This is in accordance with the condition of self-superposability laid down by Ballabh [3]. Hence the flow of viscous incompressible fluid of equal kinematics viscosity and magnetic viscosity past a perfectly conducting porous flat plate, started impulsiively from rest in the presence of transverse magnetic field is self-superposable.

It was found by Ballabh [3] that, if the axis of the symmetry in the axially symmetrical flow be x axis and the axis perpendicular to it be R-axis, The condition for self-superposability of the flow will be

$$\zeta = Rf(\Psi), \qquad ...(4.2)$$

where ζ is the vorticity of flow, $f(\Psi)$ is any function of the stream function ψ. Condition (4.2) i our case reduce to

$$\zeta = yf(\Psi). \qquad ...(4.3)$$

Since the axis perpendicular to the flow in this case has been taken as the y-axis.

Now if $\alpha = 0$ and $v_s = 0$, equation (4.3) yields for the flow as

$$f(\Psi) = \frac{U}{y\sqrt{\pi v t}} \exp\left(-\frac{y^2}{4vt}\right). \qquad ...(4.4)$$

It is now evident that for the flow (2.12) under the condition $\alpha = 0$ and $v_s = 0$, the right hand side of equation (4.4) is a function of y at any instant. It means that at any particular time the stream function of the flow can be denoted as

$$\Psi = \Psi(y). \qquad ...(4.5)$$

Thus the stream function ψ of the flow is a function of y and is in the direction of z axis i.e., in the direction perpendicular to the axis of flow and the direction of the magnetic field both.

5. Vorticity of flow. From equation (2.12) we get the vorticity of flow as

$$\zeta = \frac{U}{2\sqrt{\pi v t}}\left\{\exp\left(-\left(\frac{y + (\alpha - v_s)t}{2\sqrt{vt}}\right)^2\right) + \exp\left(-\left(\frac{y + (\alpha + v_s)t}{2\sqrt{vt}}\right)^2\right)\right\} \qquad ...(5.1)$$

Let the motion is such that

$$v_s = v \text{ and } \alpha = 2v_s = 2v$$

then,

$$\zeta = \frac{U}{2\sqrt{\pi v t}}\left\{\exp\left(-\left(\frac{y}{2\sqrt{vt}}+\frac{1}{2}\sqrt{vt}\right)^2\right)+\exp\left(-\left(\frac{y}{2\sqrt{vt}}+\frac{3}{2}\sqrt{vt}\right)^2\right)\right\} \quad \text{...(5.2)}$$

6. For Injection or Suction.

Case I. When $t=1/v$ we have

$$\frac{\zeta}{U} = \frac{U}{2\sqrt{\pi}}\left\{\exp\left(-\frac{(y+1)^2}{4}\right)+\exp\left(-\frac{(y+3)^2}{4}\right)\right\}. \quad \text{...(6.1)}$$

Case II . When $t=2/v$, we have

$$\frac{\zeta}{U} = \frac{U}{2\sqrt{2\pi}}\left\{\exp\left(-\frac{(y+2)^2}{8}\right)+\exp\left(-\frac{(y+6)^2}{8}\right)\right\}. \quad \text{...(6.2)}$$

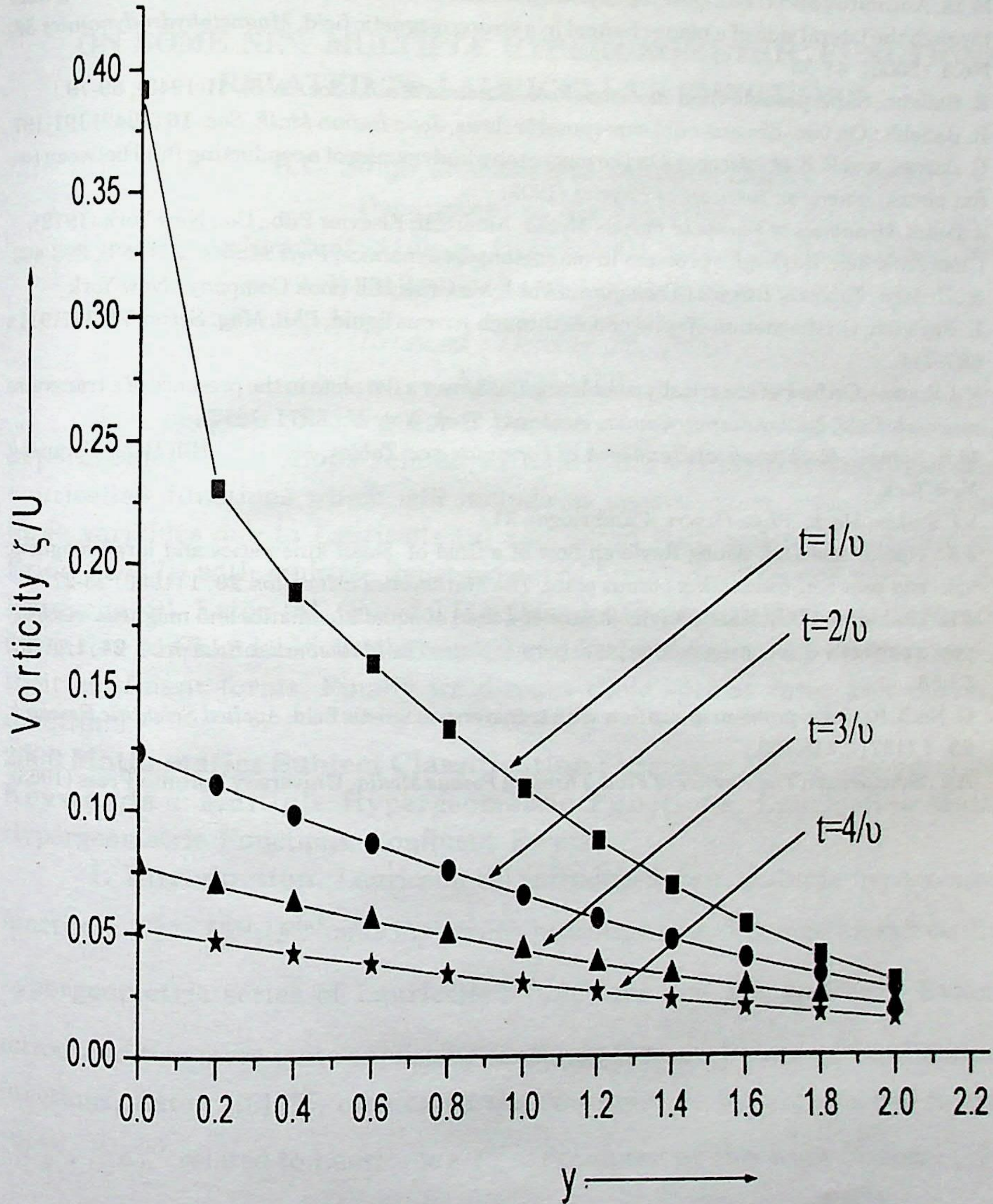
Case III . When $t=3/v$, we have

$$\frac{\zeta}{U} = \frac{U}{2\sqrt{3\pi}}\left\{\exp\left(-\frac{(y+3)^2}{12}\right)+\exp\left(-\frac{(y+9)^2}{12}\right)\right\}. \quad \text{...(6.3)}$$

Case IV . When $t= 4/v$, we have

$$\frac{\zeta}{U} = \frac{U}{4\sqrt{\pi}}\left\{\exp\left(-\frac{(y+4)^2}{16}\right)+\exp\left(-\frac{(y+12)^2}{12}\right)\right\}. \quad \text{...(6.4)}$$

7. Results and Discussion. To observe the quantitative effects on vorticity field numerical results have been calculated and plotted for above four cases. It is clear from the graph that the vorticity is maximum at the plate and it decreases as we move away from plate. At small times the vorticity near the plate falls abruptly and then it decreases and become steadier as we move far from the plate. As time increases the fall in vorticity becomes less sharp in comparison to that t=1/v. Thus as time increases the vorticity tends to become zero throughout. Thus after large time we may expect an almost irrotational flow.



Vorticity Profile for Suction or Injection

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ON SOME NEW MULTIPLE HYPERGEOMETRIC FUNCTIONS RELATED TO LAURICELLA'S FUNCTIONS

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ABSTRACT

The perpose of the present paper is to introduce some new multiple hypergeometric functions related to Lauricella's functions and intermediate Lauricella's functions which will include as special cases some of functions of three variables due to Lauricella [9] and Saran [10]; and four variables due to Exton ([5],[7]) with multiple hypergeometric functions of several variables due to Lauricella [9], Exton [6], Chandel [1], Chandel-Gupta [2], Karlsson [8], confluent forms due to Chandel-Vishwakarma [3] and Vishwakarma [12]. We also introduce their confluent forms. Finally we discuss their special cases and convergent conditons.

2000 Mathematics Subject Classification : Pramary 33C65; Secondary 33C70
Keywords : Multiple Hypergeometric Functions, Lauricellas Multiple Hypergeometric Functions, Confluent Forms.

1. Introduction. Lauricella [9] introduced four multiple hypergeometric function's $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ which bear his name. The well known confluent hypergeometric series of Lauricella's fuñctions are $\phi_2^{(n)}$ and $\psi_2^{(n)}$. Exton [7] introduced two more quite applicable confluent forms $\Xi_1^{(n)}$ and $\phi_3^{(n)}$ of Lauricella's functions. Exton ([6],[7]) considered the two multiple hypergeometric functions ${}^{(k)}_{(1)}E_D^{(n)}$, ${}^{(k)}_{(2)}E_D^{(n)}$ related to Lauricella's $F_D^{(n)}$. Prompted by this work Chandel [1] also introduced and studied the function ${}^{(k)}_{(1)}E_C^{(n)}$ related to Lauricella's $F_C^{(n)}$. Further, Chandel and Gupta [2] introduced multiple hypergeometric functions related to Lauricella's functions (later on called **Intermediate Lauricella's Functions**): ${}^{(k)}F_{AC}^{(n)}$, ${}^{(k)}F_{AD}^{(n)}$, ${}^{(k)}F_{BD}^{(n)}$ and their confluent forms ${}^{(k}_{(1)}\phi_{AC}^{(n)}$, ${}^{(k}_{(2)}\phi_{AC}^{(n)}$, ${}^{(k}_{(1)}\phi_{AD}^{(n)}$, ${}^{(k}_{(1)}\phi_{BD}^{(n)}$, ${}^{(k}_{(2)}\phi_{BD}^{(n)}$. Prompted by the above work, Karlsson [8] similarly introduced one more intermediate Lauricella's function ${}^{(k)}F_{CD}^{(n)}$. Further, Chandel-Vishwakarma ([3],[4])

introduced confluent forms ${}^{(k)}_{(1)}\phi^{(n)}_{CD}$, ${}^{(k)}_{(2)}\phi^{(n)}_{CD}$, ${}^{(k)}_{(3)}\phi^{(n)}_{CD}$, ${}^{(k)}_{(4)}\phi^{(n)}_{CD}$. Vishwakarma [12] introduced some more confluent forms of above multiple hypergeometric functions: ${}^{(k)}_{(5)}\phi^{(n)}_{CD}$, ${}^{(k)}_{(6)}\phi^{(n)}_{CD}$, ${}^{(k)}_{(2)}\phi^{(n)}_{AD}$, ${}^{(k)}_{(3)}\phi^{(n)}_{BD}$.

Motivated by above work hare in the present paper, we introduce new multiple hypergeometric functions related to Lauricella's functions and Intermeditae Lauricella's functions, which include as special cases some functions of three variables of Lauricella [9] and Saran [10] and four variables due to Exton ([5],[7]) with multiple hypergeometric functions of several variables due to Lauricella [9], Exton [6], Chandel [1], Chandel-Gupta [2], Karlsson [8], confluent forms due to Chandel- Vishwakarma [3] and Vishwakamra [12]. We also introduce their confluent forms. Finally, we discuss their special cases and convergent conditions.

2. New Functions Related to Lauricella's Functions. In this section, we introduce following three new multiple hypergeometric functions related to Lauricella's functions.

(2.1) ${}^{(k,k')}_{(1)}E^{(n)}_{D}(a,b_1,...,b_n;c,c',c'';x_1,..,x_n)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_n)(b_1,m_1)....(b_n,m_n)}{(c,m_1+...+m_k)(c',m_{k+1}+...+m_{k'})(c'',m_{k'+1}+...+m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$1 \le k \le k' \le n; k,k',n \in N$$

It is clear that

(2.1(a)) ${}^{(k,n)}_{(1)}E^{(n)}_{D}(a,b_1,...,b_n;c,c',c'';x_1,..,x_n)$

$={}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,...,b_n;c,c';x_1,..,x_n)$. (Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$ for $k'=n$)

(2.1(b)) ${}^{(k,k)}_{(1)}E^{(n)}_{D}(a,b_1,...,b_n;c,c',c'';x_1,..,x_n)$

$={}^{(k)}_{(1)}E^{(n)}_{D}(a,b_1,...,b_n;c,c'';x_1,..,x_n)$. (Exton's ${}^{(k)}_{(1)}E^{(n)}_{D}$ for $k'=k$)

(2.1(c)) ${}^{(n,n)}_{(1)}E^{(n)}_{D}(a,b_1,...,b_n;c,c',c'';x_1,..,x_n)$

$=F^{(n)}_{D}(a,b_1,...,b_n;c;x_1,...,x_n)$ (Lauricella's $F^{(n)}_{D}$, for $k'=k=n$)

(2.2) ${}^{(k,k')}_{(2)}E^{(n)}_{D}(a,a',a'',b_1,...,b_n;c;x_1,..,x_n)$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_k)(a',m_{k+1}+...+m_{k'})(a'',m_{k'+1}+...+m_n)(b_1,m_1)....(b_n,m_n)}{(c,m_1+...+m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}, \quad 1 \le k \le k' \le n; \quad k,k',n \in N.$$

(2.2.(a)) ${}^{(k,n)}_{(2)}E_D^{(n)}(a,a',a'',b_1,...,b_n;c;x_1,..,x_n)$

$={}^{(k)}_{(2)}E_D^{(n)}(a,a',b_1,...,b_n;c;x_1,..,x_n)$, (Exton's ${}^{(k)}_{(2)}E_D^{(n)}$, for $k'=n$).

(2.2(b)) ${}^{(k,k)}_{(2)}E_D^{(n)}(a,a',a'',b_1,...,b_n;c;x_1,..,x_n)$

$={}^{(k)}_{(2)}E_D^{(n)}(a,a'',b_1,...,b_n;c;x_1,..,x_n)$ (Exton's ${}^{(k)}_{(2)}E_D^{(n)}$, for $k'=k$)

(2.2(c)) ${}^{(n,n)}_{(2)}E_D^{(n)}(a,a',a'',b_1,...,b_n;c;x_1,..,x_n)$

$=F_D^{(n)}(a,b_1,...,b_n;c;x_1,...,x_n)$ (Lauricella's $F_D^{(n)}$, for $k'=k=n$).

(2.3) ${}^{(k,k')}_{(1)}E_C^{(n)}(a,a',a'',b;c_1,...,c_n;x_1,..,x_n)$

$$=\sum_{m_1,...,m_n=0}^{\infty}\frac{(a,m_1+...+m_k)(a',m_{k+1}+...+m_{k'})(a'',m_{k'+1}+...+m_n)(b,m_1+...+m_n)}{(c_1,m_1)...(c_n,m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!},\quad 1\le k\le k'\le n;\quad k,k',n\in N.$$

(2.3(a)) ${}^{(k,n)}_{(1)}E_C^{(n)}(a,a',a'',b;c_1,...,c_n;x_1,..,x_n)$

$={}^{(k)}_{(1)}E_C^{(n)}(a,a',b;c_1,...,c_n;x_1,..,x_n)$. (Chandel's ${}^{(k)}_{(1)}E_C^{(n)}$, for $k'=n$)

(2.3.(b)) ${}^{(k,k)}_{(1)}E_C^{(n)}(a,a',a'',b;c_1,...,c_n;x_1,..,x_n)$

$={}^{(k)}_{(1)}E_C^{(n)}(a,a'',b;c_1,...,c_n;x_1,..,x_n)$ (Chandel's ${}^{(k)}_{(1)}E_C^{(n)}$, for $k'=k$)

(2.3(c)) ${}^{(n,n)}_{(1)}E_C^{(n)}(a,a',a'',b;c_1,...,c_n;x_1,..,x_n)$

$=F_C^{(n)}(a,b;c_1,...,c_n;x_1,..,x_n)$ (Lauricella's $F_C^{(n)}$, for $k=k'=n$)

3. New Intermediate Lauricella's Functions. In this section, we introduce following new intermediate Lauricella's functions :

(3.1) ${}^{(k,k')}F_{AC}^{(n)}(a,b,b',b_{k'+1},...,b_n;c_1,...,c_n;x_1,..,x_n)$

$$=\sum_{m_1,...,m_n=0}^{\infty}\frac{(a,m_1+...+m_n)(b,m_1+...+m_k)(b',m_{k+1}+...+m_{k'})(b_{k'+1},m_{k'+1})...(b_n,m_n)}{(c_1,m_1)...(c_n,m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!},\quad 1\le k\le k'\le n;\quad k,k',n\in N.$$

It is clear that

$$^{(k,k)}F_{AC}^{(n)} = {}^{(k)}F_{AC}^{(n)},\quad {}^{(0,k')}F_{AC}^{(n)} = {}^{(k')}F_{AC}^{n},\quad {}^{(n,n)}F_{AC}^{(n)} = F_{C}^{(n)},\quad {}^{(1,1)}F_{AC}^{(n)} = F_{A}^{(n)},\quad {}^{(0,0)}F_{AC}^{(n)} = F_{A}^{(n)}$$

(3.2) $^{(k,k')}F_{AD}^{(n)}(a,b_1,\ldots,b_n;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b_1,m_1)\ldots(b_n m_n)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$1 \le k \le k' \le n,\ k,k',n \in N.$$

which for special value of k and k', gives

$$^{(i,k)}F_{AD}^{(n)} = {}^{(k)}F_{AD}^{(n)},\ {}^{(0,k')}F_{AD}^{(n)} = {}^{(k')}F_{AD}^{(n)},\ {}^{(n,n)}F_{AD}^{(n)} = F_{D}^{(n)},\ {}^{(1,1)}F_{BD}^{(n)} = F_{B}^{(n)},\ {}^{(0,0)}F_{BD}^{(n)} = F_{B}^{(n)},.$$

(3.3) $^{(k,k')}F_{BD}^{(n)}(a,a',a_{k'+1},\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)(a',m_{k+1}+\ldots+m_{k'})(a_{k'+1},m_{k'+1})\ldots(a_n,m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

which suggests

$$^{(k,k)}F_{BD}^{(n)} = {}^{(k)}F_{BD}^{(n)},\quad {}^{(0,k')}F_{BD}^{(n)} = {}^{(k')}F_{BD}^{(n)},\quad {}^{(n,n)}F_{BD}^{(n)} = F_{D}^{(n)},\quad {}^{(1,1)}F_{BD}^{(n)} = F_{B}^{(n)},\quad {}^{(0,0)}F_{BD}^{(n)} = F_{B}^{(n)}$$

(3.4) $^{(k,k')}F_{CD}^{(n)}(a,b,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b,m_{k+1}+\ldots+m_{k'})(b',m_{k'+1}+\ldots+m_n)(b_1,m_1)\ldots(b_k,m_k)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!},$$

so that

$$^{(k,k)}F_{CD}^{(n)} = {}^{(k)}F_{CD}^{(n)},\quad {}^{(0,k')}F_{CD}^{(n)} = {}^{(k')}F_{CD}^{(n)},\quad {}^{(n,n)}F_{CD}^{(n)} = F_{D}^{(n)},\quad {}^{(1,1)}F_{CD}^{(n)} = {}^{(1)}_{(1)}E_{C}^{(n)},\quad {}^{(0,0)}F_{CD}^{(n)} = F_{C}^{(n)}$$

4. Confluent Forms. In this section, we introduce following confluent forms of above multiple hypergeometric functions :

(4.1) $\lim_{c\to 0} {}^{(k,k')}_{(1)}E_{D}^{(n)}(a,b_1,\ldots,b_n;c,c',c'';cx_1,\ldots,cx_k,x_{k+1},\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c',m_{k+1}+\ldots+m_{k'})(c'',m_{k'+1}+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$={}^{(k,k')}_{(1)}\phi^{(n)}_{D_1}(a,b_1,..,b_n;c',c'';x_1,...,x_n).$$

Similarly,

(4.2) $${}^{(k,k')}_{(1)}\phi^{(n)}_{D_2}(a,b_1,..,b_n;c,c'';x_1,...,x_n)$$

$$=\sum_{m_1,...,m_n=0}^{\infty}\frac{(a,m_1+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_k)(c'',m_{k'+1}+...+m_n)}\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}.$$

(4.3) $${}^{(k,k')}_{(1)}\phi^{(n)}_{D_3}(a,b_1,..,b_n;c,c';x_1,...,x_n)$$

$$=\sum_{m_1,...,m_n=0}^{\infty}\frac{(a,m_1+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_k)(c',m_{k+1}+...+m_{k'})}\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}.$$

(4.4) $$\lim_{a\to\infty}{}^{(k,k')}_{(2)}E^{(n)}_{D}\left(a,a',a'',b_1,...,b_n;c;\frac{x_1}{a},...,\frac{x_k}{a},x_{k+1},...,x_n\right)$$

$$=\sum_{m_1,...,m_n=0}^{\infty}\frac{(a',m_{k+1}+...+m_{k'})(a'',m_{k'+1}+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_n)}\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}.$$

(4.5) $$\lim_{a'\to\infty}{}^{(k,k')}_{(2)}E^{(n)}_{D}\left(a,a',a'',b_1,...,b_n;c;x_1,..,x_k,\frac{x_{k+1}}{a'},...,\frac{x_{k'}}{a'},x_{k'+1},...,x_n\right)$$

$$=\sum_{m_1,...,m_n=0}^{\infty}\frac{(a,m_1+...+m_k)(a'',m_{k'+1}+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_n)}\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$={}^{(k,k')}_{(2)}\phi^{(n)}_{D_2}(a,a'',b_1,..,b_n;c;x_1,...,x_n)$$

Similarly

(4.6) $${}^{(k,k')}_{(2)}\phi^{(n)}_{D_3}(a,a',b_1,..,b_n;c;x_1,...,x_n)$$

$$=\sum_{m_1,...,m_n=0}^{\infty}\frac{(a,m_1+...+m_k)(a',m_{k+1}+...+m_{k'})(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_n)}\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}.$$

(4.7) $$\lim_{a\to\infty}{}^{(k,k')}_{(1)}E^{(n)}_{C}\left(a,a',a'',b;c_1,...,c_n;\frac{x_1}{a},..,\frac{x_k}{a},x_{k+1},...,x_n\right)$$

$$=\sum_{m_1,...,m_n=0}^{\infty}\frac{(a',m_{k+1}+...+m_{k'})(a'',m_{k'+1}+...+m_n)(b,m_1+...+m_n)}{(c_1,m_1)...(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\phi^{(n)}_{C_1}(a',a'',b;c_1,\ldots,c_n;x_1,\ldots,x_n).$$

(4.8) $$\lim_{a'\to\infty} {}^{(k,k')}_{(1)}E^{(n)}_{C}\left(a,a',a'',b;c_1,\ldots,c_n;x_1,..,x_k,\frac{x_{k+1}}{a'},\ldots,\frac{x_{k'}}{a'},x_{k'+1},\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)(a'',m_{k'+1}+\ldots+m_n)(b,m_1+\ldots+m_n)}{(c_1,m_1)\ldots(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\phi^{(n)}_{C_2}(a,a'',b_1,..,b_n;c;x_1,\ldots,x_n).$$

(4.9) $$\lim_{a''\to\infty} {}^{(k,k')}_{(1)}E^{(n)}_{C}\left(a,a',a'',b;c_1,\ldots,c_n;x_1,..,x_k,x_{k+1},\ldots,x_{k'},\frac{x_{k'+1}}{a''},\ldots,\frac{x_n}{a''}\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)(a',m_{k+1}+\ldots+m_{k'})(b,m_1+\ldots+m_n)}{(c_1,m_1)\ldots(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\phi^{(n)}_{C_3}(a,a',b;c_1,\ldots,c_n;x_1,\ldots,x_n)$$

(4.10) $$\lim_{b\to\infty} {}^{(k,k')}F^{(n)}_{AC}\left(a,b,b',b_{k'+1},\ldots,b_n;c_1,\ldots,c_n;\frac{x_1}{b},..,\frac{x_k}{b},x_{k+1},\ldots,x_{k'},x_{k'+1},\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b',m_{k+1}+\ldots+m_{k'})(b_{k'+1},m_{k'+1})\ldots(b_n,m_n)}{(c_1,m_1)\ldots(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\phi^{(n)}_{AC}(a,b',b_{k'+1},..,b_n;c_1,\ldots,c_n;x_1,\ldots,x_{n'}).$$

(4.11) $$\lim_{b'\to\infty} {}^{(k,k')}F^{(n)}_{AC}\left(a,b,b',b_{k'+1},\ldots,b_n;c_1,\ldots,c_n;x_1,..,x_k,\frac{x_{k+1}}{b'},\ldots,\frac{x_{k'}}{b'},x_{k'+1},\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b,m_1+\ldots+m_k)(b_{k'+1},m_{k'+1})\ldots(b_n,m_n)}{(c_1,m_1)\ldots(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(2)}\phi^{(n)}_{AC}(a,b,b_{k'+1},..,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n).$$

(4.12) $$\lim_{b_{k'+1},\ldots,b_n\to\infty} {}^{(k,k')}F^{(n)}_{AC}\left(a,b,b',b_{k'+1},\ldots,b_n;c_1,\ldots,c_n;x_1,..,x_{k'},\frac{x_{k'+1}}{b_{k'+1}},\ldots,\frac{x_n}{b_n}\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)(b,m_1+\ldots+m_k)(b',m_{k+1}+\ldots+m_{k'})}{(c_1,m_1)\ldots(c_n,m_n)}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(3)}\phi^{(n)}_{AC}(a,b,b',..,b_n;c_1,...,c_n;x_1,...,x_n)$$

Similarly

(4.13) $${}^{(k,k')}_{(4)}\phi^{(n)}_{AC}(a,b';c_1,...,c_n;x_1,...,x_n)$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_n)(b',m_{k+1}+...+m_{k'})}{(c_1,m_1)...(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

(4.14) $$\lim_{b_1,...,b_n\to\infty} {}^{(k,k')}F^{(n)}_{AD}\left(a,b_1,...,b_n;c,c',c_{k'+1},...,c_n;\frac{x_1}{b_1},...,\frac{x_n}{b_n}\right)$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_n)}{(c,m_1+...+m_k)(c',m_{k+1}+...+m_{k'})(c_{k'+1},m_{k'+1})...(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\phi^{(n)}_{AD}(a;c,c',c_{k'+1},...,c_n;x_1,...,x_n)\ .$$

(4.15) $$\lim_{c\to 0} {}^{(k,k')}F^{(n)}_{AD}(a,b_1,...,b_n;c,c',c_{k'+1},...,c_n;cx_1,...,cx_k,x_{k+1},...,x_n)$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c',m_{k+1}+...+m_{k'})(c_{k'+1},m_{k'+1})...(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(2)}\phi^{(n)}_{AD}(a,b_1,...,b_n;c',c_{k'+1},...,c_n;x_1,...,x_n).$$

(4.16) $$\lim_{c'\to 0} {}^{(k,k')}F^{(n)}_{AD}(a,b_1,...,b_n;c,c',c_{k'+1},...,c_n;x_1,...,x_k,c'x_{k+1},...,c'x_{k'},x_{k'+1},...,x_n)$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_n)(b_1,m_1)...(b_n,m_n)}{(c,m_1+...+m_k)(c_{k'+1},m_{k'+1})...(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(3)}\phi^{(n)}_{AD}(a,b_1,...,b_n;c,c_{k'+1},...,c_n;x_1,...,x_n).$$

(4.17) $$\lim_{b_1,...,b_n\to\infty} {}^{(k,k')}F^{(n)}_{BD}\left(a,a',a_{k'+1},....,a_n,b_1,...,b_n;c;\frac{x_1}{b_1},...,\frac{x_n}{b_n}\right)$$

$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a,m_1+...+m_k)(a',m_{k+1}+...+m_{k'})(a_{k'+1},m_{k'+1})...(a_n,m_n)}{(c,m_1+...+m_n)} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\phi^{(n)}_{BD}(a,a',a_{k'+1},...,a_n;c;x_1,...,x_n).$$

(4.18) $$\lim_{a\to\infty} {}^{(k,k')}F^{(n)}_{BD}\left(a,a',a_{k'+1},....,a_n,b_1,...,b_n,c;\frac{x_1}{a},...,\frac{x_k}{a},x_{k+1},...,x_n\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a',m_{k+1}+\dots+m_{k'})(a_{k'+1},m_{k'+1})\dots(a_n,m_n)(b_1,m_1),\dots,(b_n,m_n)}{(c,m_1+\dots+m_n)} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(2)}\phi^{(n)}_{BD}(a',a_{k'+1},\dots,a_n,b_1,\dots,b_n;c;x_1,\dots,x_n).$$

(4.19) $$\lim_{a'\to\infty} {}^{(k,k')}F^{(n)}_{BD}\left(a,a',a_{k'+1},\dots,a_n,b_1,\dots,b_n;c;x_1,\dots,x_k,\frac{x_{k+1}}{a'},\dots,\frac{x_{k'}}{a'},x_{k'+1},\dots,x_n\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a,m_1+\dots+m_k)(a_{k'+1},m_{k'+1})\dots(a_n,m_n)(b_1,m_1),\dots,(b_n,m_n)}{(c,m_1+\dots+m_n)} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(3)}\phi^{(n)}_{BD}(a,a_{k'+1},\dots,a_n,b_1,\dots,b_n;c;x_1,\dots,x_n).$$

(4.20) $$\lim_{a_{k'+1},\dots,a_n\to\infty} {}^{(k,k')}F^{(n)}_{BD}\left(a,a',a_{k'+1},\dots,a_n,b_1,\dots,b_n;c;x_1,\dots,x_k,x_{k+1},\dots,x_{k'},\frac{x_{k'+1}}{a_{k'+1}},\dots,\frac{x_n}{a_n}\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a,m_1+\dots+m_k)(a',m_{k+1}+\dots+m_{k'})(b_1,m_1),\dots,(b_n,m_n)}{(c,m_1+\dots+m_n)} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(4)}\phi^{(n)}_{BD}(a,a',b_1,\dots,b_n;c;x_1,\dots,x_n).$$

(4.21) $$\lim_{a\to\infty} {}^{(k,k')}F^{(n)}_{CD}\left(a,b,b',b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;\frac{x_1}{a},\dots,\frac{x_n}{a}\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(b,m_{k+1}+\dots+m_{k'})(b',m_{k'+1}+\dots+m_n)(b_1,m_1),\dots,(b_k,m_k)}{(c,m_1+\dots+m_k)(c',m_{k+1}+\dots+m_{k'})(c_{k'+1},m_{k'+1})\dots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(1)}\phi^{(n)}_{CD}(b,b',b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;x_1,\dots,x_n).$$

(4.22) $$\lim_{b\to\infty} {}^{(k,k')}F^{(n)}_{CD}\left(a,b,b',b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;x_1,\dots,x_k,\frac{x_{k+1}}{b},\dots,\frac{x_{k'}}{b},x_{k'+1},\dots,x_n\right)$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(a,m_1+\dots+m_n)(b',m_{k'+1}+\dots+m_n)(b_1,m_1),\dots,(b_k,m_k)}{(c,m_1+\dots+m_k)(c',m_{k+1}+\dots+m_{k'})(c_{k'+1},m_{k'+1})\dots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(2)}\phi^{(n)}_{CD}(a,b',b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;x_1,\dots,x_n).$$

(4.23) $$\lim_{b'\to\infty} {}^{(k,k')}F^{(n)}_{CD}\left(a,b,b',b_1,\dots,b_k;c,c',c_{k'+1},\dots,c_n;x_1,\dots,x_{k'},\frac{x_{k'+1}}{b'},\dots,\frac{x_n}{b'}\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b,m_{k+1}+\ldots+m_{k'})(b_1,m_1),\ldots,(b_k,m_k)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(3)}\phi^{(n)}_{CD}(a,b,b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_n).$$

(4.24) $$\lim_{b_1,\ldots,b_k\to\infty} {}^{(k,k')}F^{(n)}_{CD}\left(a,b,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;\frac{x_1}{b_1},\ldots,\frac{x_{k'}}{b_k},x_{k+1},\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b,m_{k+1}+\ldots+m_{k'})(b',m_{k'+1}+\ldots+m_n)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(4)}\phi^{(n)}_{CD}(a,b,b';c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_n).$$

(4.25) $$\lim_{b_1,b',b_1,\ldots,b_k\to\infty} {}^{(k,k')}F^{(n)}_{CD}\left(a,b,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;\frac{x_1}{b_1},\ldots,\frac{x_k}{b_k},\frac{x_{k+1}}{b},\ldots,\frac{x_{k'}}{b'},\ldots,\frac{x_n}{b'}\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(5)}\phi^{(n)}_{CD}(a;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_n).$$

(4.26) $$\lim_{c\to\infty} {}^{(k,k')}F^{(n)}_{CD}(a,b,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;cx_1,\ldots,cx_k,x_{k+1},\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b,m_{k+1}+\ldots+m_{k'})(b',m_{k'+1}+\ldots+m_n)(b_1,m_1),\ldots,(b_k,m_k)}{(c',m_{k+1}+\ldots+m_{k'})(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(6)}\phi^{(n)}_{CD}(a;b,b',b_1,\ldots,b_k;c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_n).$$

(4.27) $$\lim_{c'\to 0} {}^{(k,k')}F^{(n)}_{CD}(a,b,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_k,c'x_{k+1},\ldots,c'x_{k'},x_{k'+1},\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b,m_{k+1}+\ldots+m_{k'})(b',m_{k'+1}+\ldots+m_n)(b_1,m_1),\ldots,(b_k,m_k)}{(c,m_1+\ldots+m_k)(c_{k'+1},m_{k'+1})\ldots(c_n,m_n)}$$

$$\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(7)}\phi^{(n)}_{CD}(a;b,b',b_1,\ldots,b_k;c,c_{k'+1},\ldots,c_n;x_1,\ldots,x_n).$$

(4.28) $$\lim_{c_{k'+1},\ldots,c_n\to 0} {}^{(k,k')}F^{(n)}_{CD}(a,b,b',b_1,\ldots,b_k;c,c',c_{k'+1},\ldots,c_n;x_1,\ldots,x_{k'},c_{k'+1}x_{k'+1},\ldots,c_nx_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b,m_{k+1}+\ldots+m_{k'})(b',m_{k'+1}+\ldots+m_n)(b_1,m_1),\ldots,(b_k,m_k)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})}$$

$$\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k,k')}_{(8)}\phi^{(n)}_{CD}(a;b,b',b_1,\ldots,b_k;c,c';x_1,\ldots,x_n).$$

No doubt all functions studied above are included in generalized multiple hypergeometric function of several variables due to Srivastava and Daoust [11], but they have their own interest.

5. Special Cases. When $n=4$.

${}^{(k,k')}_{(1)}E^{(n)}_D$

(5.1) $${}^{(3,4)}_{(1)}E^{(4)}_D(a,b_1,b_2,b_3,b_4;c,c',-;x_1,x_2,x_3,x_4)$$

$$= K_{11}(a,a,a,a,b_1,b_2,b_3,b_4;c,c,c,c;x_1,x_2,x_3,x_4)$$ (Exton[7] p. 78, (3.3.11))

(5.2) $${}^{(2,4)}_{(1)}E^{(4)}_D(a,b_1,b_2,b_3,b_4;c,c',-;x_1,x_2,x_3,x_4)$$

$$= K_{12}(a,a_2,a,a,b_1,b_2,b_3,b_4;c,c,c',c';x_1,x_2,x_3,x_4)$$ (Exton [7] p. 78,(3.3.12))

(5.3). $${}^{(2,3)}_{(1)}E^{(4)}_D(a,b_1,b_2,b_3,b_4;c,c',c'';x_1,x_2,x_3,x_4)$$

$$= K_{13}(a,a,a,a,b_1,b_2,b_3,b_4;c,,c',c'';x_1,x_2,x_3,x_4)$$ (Exton [7] p. 78, (3.3.13))

${}^{(k,k')}_{(2)}E^{(n)}_D$

(5.4) $${}^{(3,4)}_{(2)}E^{(4)}_D(a,a',-,b_1,b_2,b_3,b_4;c;x_1,x_2,x_3,x_4)$$

$$= K_{15}(a,a,a,a',b_1,b_2,b_3,b_4;c,c,c,c;x_1,x_2,x_3,x_4)$$ (Exton [7] p. 78, (3.3.15))

(5.5) $${}^{(2,4)}_{(2)}E^{(4)}_D(a,a',b_1,b_2,b_3,b_4;c;x_1,x_2,x_3,x_4)$$

$$= K_{20}(a,a,b_3,b_4;b_1,b_2,a',a';c,c,c,c;x_1,x_2,x_3,x_4)$$ (Exton [7] p. 78, (3.3.20))

(5.6) $${}^{(2,3)}_{(2)}E^{(4)}_D(a,a',a'',b_1,b_2,b_3,b_4;c;x_1,x_2,x_3,x_4)$$

$$= K_{21}(a,a,a',a'';b_1,b_2,b_3,b_4;c,c,c,c;x_1,x_2,x_3,x_4)$$ (Exton [7] p.78, (3.3.21))

${}^{(k,k')}_{(1)}E^{(n)}_C$

(5.7) $\,^{(3,4)}_{(1)}E_C^{(4)}(a,a',-,b;c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_2(b,b,b,b;a,a,a,a';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$(Exton [7] p.78, (3.3.2)).

(5.8) $\,^{(2,4)}_{(1)}E_C^{(4)}(a,a',-,b;c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_5(b,b,b,b;a,a,a',a';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$(Exton [7] p.78, (3.3.5)).

(5.9) $\,^{(2,3)}_{(1)}E_C^{(4)}(a,a',a'',b;c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_{10}(b,b,b,b;a,a,a',a'';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$(Exton [7] p.78, (3.3.10))

$^{(k,k')}F_{AC}^{(n)}$

(5.10) $^{(3,4)}F_{AC}^{(4)}(a,b,b',b'';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_2(a,a,a,a;b,b,b,b';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$[Exton [7], p.78, (3.3.2)]

(5.11) $^{(2,3)}F_{AC}^{(4)}(a,b,b',b'';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_{10}(a,a,a,a;b,b,b',b'';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$(Exton[7]p.78, (3.3.10)).

(5.12) $^{(2,4)}F_{AC}^{(4)}(a,b,b',-;c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$

$= K_5(a,a,a,a;b,b';c_1,c_2,c_3,c_4;x_1,x_2,x_3,x_4)$ (Exton [7] p.78, (3.3.5)).

$^{(k,k')}F_{AD}^{(n)}$

(5.13) $^{(3,4)}F_{AD}^{(4)}(a,b_1,b_2,b_4;c,c',-;x,x_2,x_3,x_4)$

$= K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c';x_1,x_2,x_3,x_4)$(Exton [7] p.78, (3.3.11)).

(5.14) $^{(2,4)}F_{AD}^{(4)}(a,b_1,b_2,b_3,b_4;c,c',-;x_1,x_2,x_3,x_4)$

$= K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c',c';x_1,x_2,x_3,x_4)$(Exton [7] p.78,(3.3.12)).

(5.15) $^{(2,3)}F_{AD}^{(4)}(a,b_1,b_2,b_3,b_4;c,c',c'';x_1,x_2,x_3,x_4)$

$= K_{13}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c',c'';x_1,x_2,x_3,x_4)$(Exton [7] p.78,(3.3.13)).

$^{(k,k')}F_{BD}^{(n)}$

(5.16) $^{(3,4)}F_{BD}^{(4)}(a,a',-;b_1,b_2,b_3,b_4;c;x,x_2,x_3,x_4)$

$= K_{15}(a,a,a,a';b_1,b_2,b_3,b_4;c,c,c,c;x_1,x_2,x_3,x_4)$(Exton [7] p.78,(3.3.15)).

(5.17) $^{(2,4)}F_{BD}^{(4)}(a,a',-,b_1,b_2,b_3,b_4;c;x,x_2,x_3,x_4)$

$$= K_{20}(a,a,b_3,b_4;b_1,b_2,a',a';c,c,c,c;x_1,x_2,x_3,x_4)$$ (Exton [7] p.78,(3.3.20)).

(5.18) $^{(2,3)}F_{BD}^{(4)}(a,a',a'',b_1,b_2,b_3,b_4;c;x,x_2,x_3,x_4)$

$$= K_{21}(a,a,a',a'';b_1,b_2,b_3,b_4;c,c,c,c;x_1,x_2,x_3,x_4)$$ (Exton [7] p.78,(3.3.21)).

$^{(k,k')}F_{CD}^{(n)}$

(5.19) $^{(2,3)}F_{CD}^{(4)}(a,b,b',b_1,b_2;c,c',c_4;x,x_2,x_3,x_4)$

$$= K_{13}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c',c_4;x_1,x_2,x_3,x_4),$$ (Exton[7] p.78,(3.3.13)).

(5.20) $^{(2,2)}F_{CD}^{(4)}(a,-,b',b_1,b_2;c,-,c_3,c_4;x_1,x_2,x_3,x_4)$

$$= K_{12}(a,a,a,a;b_1,b_2,b',b';c,c,c_3,c_4;x_1,x_2,x_3,x_4),$$ (Exton [7] p.78,(3.3.12)).

(5.21) $^{(3,4)}F_{CD}^{(4)}(a,b,-,b_1,b_2,b_3;c,c',-;x_1,x_2,x_3,x_4)$

$$= K_{11}(a,a,a,a;b_1,b_2,b_3,b;c,c,c,c';x_1,x_2,x_3,x_4),$$ (Exton [7] p.78,(3.3.11)).

(5.22) $^{(1,2)}\phi_{CD}^{(4)}(a,b,b',b_1;c,c',c_3,c_4;x_1,x_2,x_3,x_4)$

$$= K_{10}(a,a,a,a;b',b',b_1,b;c_3,c_4,c,c';x_3,x_4,x_1,x_2),$$ (Exton [7] p.78,(3.3.10)).

(5.23) $^{(2,2)}F_{CD}^{(4)}(a,-,b',b_1,b_2;c,-,c_3,c_4;x_1,x_2,x_3,x_4)$

$$= K_9(a,a,a,a;b',b',b_1,b_2;c_3,c_4,c,c;x_3,x_4,x_1,x_2),$$ (Exton [7],p.78(3.3.9)).

6. Special Cases. When $n=3$.

$^{(k,k')}_{(1)}E_D^{(n)}$

(6.1) $^{(1,3)}_{(1)}E_D^{(3)}(a,b_1,b_2,b_3;c,c',-;x_1,x_2,x_3)$

$$= F_G(a,a,a,b_1,b_2,b_3;c,c',c';x_1,x_2,x_3) \quad r_1+r_2=1,\ \ r_1+r_3=1.$$

(6.2) $^{(1,2)}_{(1)}E_D^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3)$

$$= F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3) \quad |x_1|+|x_2|+|x_3|<1.$$

$^{(k,k')}_{(2)}E_D^{(n)}$

(6.3) $^{(1,3)}_{(2)}E_D^{(3)}(a,a',-,b_1,b_2,b_3;c;x_1,x_2,x_3)$

$$= F_3(a,a',a',b_1,b_2,b_3;c,c,c;x_1,x_2,x_3), \quad r_1+r_2=r_1r_2,\ \ r_2=r_3.$$

(6.4) $^{(1,2)}_{(2)}E_D^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3;c;x_1,x_2,x_3)$

$$= F_B^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3;c;x_1,x_2,x_3), \quad |x_1|<1,|x_2|<1,|x_3|<1.$$

(6.5) $\quad {}^{(1,2)}_{(1)}E_C^{(3)}(a_1,a_2,a_3,b;c_1,c_2,c_3;x_1,x_2,x_3)$

$$= F_A^{(3)}(b,a_1,a_2,a_3;c_1,c_2,c_3;x_1,x_2,x_3) \quad |x_1|+|x_2|+|x_3|<1 .$$

(6.6) $\quad {}^{(1,3)}_{(1)}E_C^{(3)}(a_1,a_2,-,b;c_1,c_2,c_3;x_1,x_2,x_3)$

$$= F_E(b,b,b,a_1,a_2,a_2;c_1,c_2,c_3;x_1,x_2,x_3) \quad |r_1|+\left(\sqrt{r_2}+\sqrt{r_3}\right)^2=1 .$$

${}^{(k,k')}F_{AC}^{(n)}$

(6.7) $\quad {}^{(1,3)}F_{AC}^{(3)}(a,b,b',-;c_1,c_2,c_3;x_1,x_2,x_3)$

$$= F_E(a,a,a,b,b',b';c_1,c_2,c_3;x_1,x_2,x_3), \quad |r_1|+\left(\sqrt{r_2}+\sqrt{r_3}\right)^2=1 .$$

(6.8) $\quad {}^{(1,2)}F_{AC}^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3)$

$$= F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3), \quad |x_1|+|x_2|+|x_3|<1 .$$

${}^{(k,k')}F_{AD}^{(n)}$

(6.9) $\quad {}^{(1,1)}F_{AD}^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3)$

$$= F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3), \quad |x_1|+|x_2|+|x_3|<1 .$$

(6.10) $\quad {}^{(1,3)}F_{AD}^{(3)}(a,b_1,b_2,b_3;c,c',-;x_1,x_2,x_3)$

$$= F_G(a,a,a,b_1,b_2,b_3;c,c',c';x_1,x_2,x_3), \quad r_1+r_2=1, r_1+r_3=1 .$$

${}^{(k,k')}F_{BD}^{(n)}$

(6.11) $\quad {}^{(1,2)}F_{BD}^{(3)}(a_1,a_2,a_3;b_1,b_2,b_3;c;x_1,x_2,x_3)$

$$= F_B^{(3)}(a_1,a_2,a_3,b_1,b_2,b_3;c;x_1,x_2,x_3), \quad |x_1|<1,|x_2|<1,|x_3|<1 .$$

(6.12) $\quad {}^{(1,3)}F_{BD}^{(3)}(a_1,a_2,-,b_1,b_2,b_3;c;x_1,x_2,x_3)$

$$= F_S(a_1,a_2,a_2,b_1,b_2,b_3;c\ ,c\ ,c\ ;x_1,x_2,x_3) \quad r_1+r_2=r_1r_2,\ r_1=r_3 .$$

${}^{(k,k')}F_{CD}^{(n)}$

(6.13) $\quad {}^{(1,2)}F_{CD}^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3)$

$$= F_A^{(3)}(a,b_1,b_2,b_3;c_1,c_2,c_3;x_1,x_2,x_3), \quad |x_1|+|x_2|+|x_3|<1 .$$

(6.14) $\quad {}^{(1,1)}F_{CD}^{(3)}(a,-,b',b;c_1,c_2,c_3;x_1,x_2,x_3)$

$$= F_E(a,a,a,b,b',b';c_1,c_2,c_3;x_1,x_2,x_3),\ r_1+\left(\sqrt{r_2}+\sqrt{r_3}\right)^2=1.$$

Here $F_A^{(3)}, F_B^{(3)}$ are Lauricella's series [14], while F_G, F_S, F_E are hypergeometric series of three variables defined by Saran [16] already Conjectured by Lauricella [14] and r_1, r_2, r_3 are associated radii of convergence of the triple hypergeometric series.

7. Convergence Conditions.

7.1 Convergence Conditions for ${}^{(k,k')}_{(1)}E_D^{(n)}$.

For ${}^{(k,k')}_{(1)}E_D^{(n)}$,

$$A_{m_1,\ldots,m_n}=\frac{(a_1,m_1+\ldots+m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})(c'',m_{k'+1}+\ldots+m_n)}\frac{1}{m_1!}\cdots\frac{1}{m_n!},$$

$$1\le k\le k'\le n.$$

Therefore

$$f_i(m_1,\ldots,m_n)=\frac{A_{m_1,\ldots,m_{i-1},m_i+1,m_{i+1},\ldots,m_n}}{A_{m_1,\ldots,m_n}}$$

$$=\frac{\left[\begin{array}{c}\dfrac{(a,m_1+\ldots+m_{i-1}+m_i+1+m_{i+1}+\ldots+m_k+\ldots+m_n)}{(c,m_1+\ldots+m_{i-1}+m_i+1+m_{i+1}+\ldots+m_k)(\ldots)}\\ \dfrac{(b_1,m_1)\ldots(b_{i-1},m_{i-1})(b_i,m_i+1)(b_{i+1},m_{i+1})\ldots(b_n,m_n)}{(\ldots)m_1!\ldots m_{i-1}!(m_i+1)!m_{i+1}!\ldots m_n!}\end{array}\right]}{\dfrac{(a,m_1+\ldots+m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_{k'})(c'',m_{k'+1}+\ldots+m_n)}\dfrac{1}{m_1!}\cdots\dfrac{1}{m_n!}}$$

$$=\frac{(a+m_1+\ldots+m_n)(b_{m_i}+m_i)}{(c+m_1+\ldots+m_i+\ldots+m_k)(m_i+1)}$$

Hence

$$\phi_i(m_1,\ldots,m_n)=\lim_{\in\to\infty}\frac{a+\in(m_1+\ldots+m_n)(b_{m_i}+m_i\in)}{(c+\in(m_1+\ldots+m_i+\ldots+m_k))(m_i\in+1)}$$

$$=\frac{(m_1+\ldots+m_n)m_i}{(m_1+\ldots+m_i+\ldots+m_k)m_i}.$$

Therefore, $r_i=\dfrac{1}{\phi_i(m_1,\ldots,m_n)}=\dfrac{m_1+\ldots+m_k}{m_1+\ldots+m_n}$ (which is independent of i)

Hence $r_i = r_1 = \ldots = r_k$. $(1 \le i \le k)$

Similarly,

$$r_i = \frac{m_{k+1} + \ldots + m_{k'}}{m_1 + \ldots + m_n} \quad \text{(indpendent of } i\text{)}$$

$$= r_{k+1} = \ldots = r_{k'} \quad (k \le i \le k')$$

Also $$r_i = \frac{m_{k'+1} + \ldots + m_n}{m_1 + \ldots + m_n} \quad \text{(Independent of } i\text{)}$$

$$= r_{k'+1} = \ldots = r_n \quad (k' \le i \le n)$$

Thus $$r_k + r_{k'} + r_n = \frac{(m_1 + \ldots + m_k) + (m_{k+1} + \ldots + m_{k'}) + (m_{k'+1} + \ldots + m_n)}{m_1 + \ldots + m_n}$$

$= 1$. (**Required condition**)

where $r_1, \ldots, r_k, \ldots, r_{k'}, \ldots, r_n$ are associted radi of convergence.

Similarly required ***Codition for Convergence of*** ${}^{(k,k')}_{(2)}E_D^{(n)}$ is given by

(7.2) $$\frac{1}{r_k} + \frac{1}{r_{k'}} + \frac{1}{r_n} = 1.$$

Convergence conditon for ${}^{(k,k')}_{(1)}F_C^{(n)}$ is given by

(7.3) $$\left(\sqrt{r_1} + \ldots + \sqrt{r_k}\right)^2 + \left(\sqrt{r_{k+1}} + \ldots + \sqrt{r_{k'}}\right)^2 + \left(\sqrt{r_{k'+1}} + \ldots + \sqrt{r_n}\right)^2 = 1\,.$$

Convergence conditon for ${}^{(k,k')}F_{AC}^{(n)}$ is given by

(7.4) $$\left(\frac{1}{\sqrt{r_1}} + \ldots + \frac{1}{\sqrt{r_k}}\right)^2 + \left(\frac{1}{\sqrt{r_{k+1}}} + \ldots + \frac{1}{\sqrt{r_{k'}}}\right)^2 + \frac{1}{r_{k'+1}} + \ldots + \frac{1}{r_n} = 1.$$

Convergence conditon for ${}^{(k,k')}F_{AD}^{(n)}$ is given by

(7.5) $$r_k + r_{k'} + r_{k'+1} + \ldots + r_n = 1.$$

Convergence conditon for ${}^{(k,k')}F_{BD}^{(n)}$ is given by

(7.6) $$\frac{1}{r_k} + \frac{1}{r_{k'}} + \frac{1}{r_{k'+1}} + \ldots + \frac{1}{r_n} = 1,$$

Also **Convergence conditon for** ${}^{(k,k')}F_{CD}^{(n)}$ is given by

$$(7.7) \quad r_k + r_{k+1} + \ldots + r_{k'} + \left(\sqrt{r_{k+1}} + \ldots + \sqrt{r_n}\right)^2 = 1.$$

We can also derive (i) Fractional Derivatives (ii) Fractional Integration and (iii) Analytic continuation and multidimensional integral transforms etc. Also applications of these functions can be shown in (i) heat conduction problems, Electrostatic Potential problems and in other similar problems of physical sciences.

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CERTAIN MULTIPLE INTEGRAL TRANSFORMATIONS PERTANING TO THE MULTIVARIABLE A-FUNCTION

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ABSTRACT

The object of this paper is to establish two general multiple integral transformations of the multivariable A-function (1981), as a Kernel product with Fox's H-function [3,p. 408] and Laguerre polynomials respectively with the gernal class of polynomials ([4] and [7]). Several possible cases are also included.

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Keywords: Multivariable A-function, Multivariable H-function, Hermite polynomials, Laguerre polynomails, Jacobi polynomials.

1. Introduction. Gautam and Goyal (1981) defined the multivariable A-function which is generalization of multivariable H-function of Srivastava and Panda [6]. The difinition of multivariable A-function runs as follows:

$$A[z_1,\ldots,z_r] = A^{\mu,\lambda:\mu_1,\lambda_1;\ldots;\mu_r,\lambda_r}_{\upsilon,C:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}\left[\begin{matrix} z_1 \\ z_r \end{matrix}\middle| \begin{matrix} (a_j;A_j;\ldots;A_j^{(r)})_{1,\upsilon};(\tau_j,C_j)_{1,c_1};\ldots;(\tau_j^{(r)},C_j^{(r)})_{1,c_r} \\ (b_j;B_j;\ldots;B_j^{(r)})_{1,c};(d_j,D_j)_{1,c_1};\ldots;(d_j^{(r)},D_j^{(r)})_{1,c_r} \end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \theta_1(s_1)\cdots\theta_r(s_r)\Phi(s_1,\ldots,s_r) z_1^{s_1}\cdots z_r^{s_r}\, ds_1\cdots ds_r, \qquad \ldots(1.1)$$

where $\omega = \sqrt{-1}$

$$\theta_i(s_i) = \frac{\prod_{j=1}^{\mu_i}\Gamma\left(d_j^{(i)} - D_j^{(i)}s_i\right)\prod_{j=1}^{\mu_i}\Gamma\left(1-\tau_j^{(i)} + C_j^{(i)}s_i\right)}{\prod_{j=\mu_i+1}^{C_1}\Gamma\left(1-d_j^{(i)} + D_j^{(i)}s_i\right)\prod_{j=\lambda_1+1}^{\upsilon_1}\Gamma\left(\tau_j^{(i)} - C_j^{(i)}s_i\right)}, \quad \forall i = 1,\ldots,r, \qquad \ldots(1.2)$$

$$\Phi(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{\lambda}\Gamma\left(1-a_j + \sum_{i=1}^{r} A_j^{(i)}s_i\right)\prod_{j=1}^{\mu}\Gamma\left(bj - \sum_{i=1}^{r} B_j^{(i)}s_i\right)}{\prod_{j=\lambda+1}^{\upsilon}\Gamma\left(a_j - \sum_{i=1}^{r} A_j^{(i)}s_i\right)\prod_{j=\mu+1}^{C}\Gamma\left(1-b_j + \sum_{i=1}^{r} B_j^{(i)}s_i\right)} \qquad \ldots(1.3)$$

Here $\mu,\lambda,\upsilon,C,\mu_i,\lambda_i,\upsilon_i$ and c_i are non-negative integers and all $a_j, b_j, d_j^{(i)}, \tau_j^{(i)}, B_j^{(i)}$, are complex numbers. The multiple integral defining the A-functon of r-variables converges absolutely if

$$\xi_i^* = 0, \quad \text{...(1.4)}$$

$$\eta_i > 0, \quad \text{...(1.5)}$$

$$\text{and } |\arg(\xi_i) z_k| < \eta\pi/2, \quad \text{...(1.6)}$$

where

$$\xi_i = \prod_{J=1}^{\upsilon}\left\{A_j^{(i)}\right\}^{A_j^{(i)}} \prod_{J=1}^{C}\left\{B_j^{(i)}\right\}^{-B_j^{(i)}} \prod_{J=1}^{c_i}\left\{D_j^{(i)}\right\}^{D_j^{(i)}} \prod_{J=1}^{\upsilon_i}\left\{C_j^{(i)}\right\}^{-C_j^{(i)}}, \quad \text{...(1.7)}$$

$$\xi_i^* = img\left[\sum_{j=1}^{\upsilon} A_j^{(i)} - \sum_{j=1}^{C} B_j^{(i)} + \sum_{j=1}^{c_i} D_j^{(i)} - \sum_{j=1}^{\upsilon_i} C_j^{(i)}\right], \quad \text{...(1.8)}$$

$$\eta_i = \text{Re}\left[\sum_{j=1}^{\lambda} A_j^{(i)} - \sum_{j=1}^{\upsilon} A_j^{(i)} + \sum_{j=1}^{\mu} B_j^{(i)} - \sum_{j=1}^{C} B_j^{(i)} + \sum_{j=1}^{\mu_i} D_j^{(i)} - \sum_{j=1}^{c_i} D_j^{(i)} + \sum_{j=1}^{\lambda_i} C_j^{(i)} - \sum_{j=1}^{\upsilon_i} C_j^{(i)}\right]$$

$$\forall i = 1,\ldots,r. \quad \text{...(1.9)}$$

If we take all $A_j^{(i)}s, B_j^{(i)}s, C_j^{(i)}s$, and $D_j^{(i)}$ as real and $\mu = 0$, the A-function reduces to multivariable H-function of Srivastava and Panda [1976 b].

Srivastava[4] introduced the general class of polynomials (see also Srivastava and Singh[7])

$$S_\alpha^\beta[z] = \sum_{k=0}^{[\beta/\alpha]} \frac{(-\beta)_{k\alpha}}{k!} A_{\beta,k} z^k, \beta = 0,1,2,\ldots, \quad \text{...(1.10)}$$

where α is an arbitrary positive integer and coefficient $A_{\beta,k}(\beta, k \geq 0)$ are arbitary constant, real of complex.

2. The Main Results.

(i) $$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1 - 1} \cdots x_r^{\sigma_r - 1}\left(k_1 x_1^{\rho_1} + \cdots + k_r x_r^{\rho_r}\right)^\sigma S_\alpha^\beta\left[\eta\left(k_1 x_1^{\rho_1} + \cdots + k_r x_r^{\rho_r}\right)^h\right]$$

$$H_{p,q}^{m,0}\left[\xi\left(k_1 x_1^{\rho_1} + \cdots + k_r x_r^{\rho_r}\right)\Big|_{(g_1,\gamma_1),\ldots,(g_q,\gamma_q)}^{(e_1,\varepsilon_1),\ldots,(e_p,\varepsilon_p)}\right]$$

$$A^{\mu,\lambda:\mu_1,\ldots;\mu_r,\lambda_r}_{\upsilon,C:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}\begin{bmatrix} z_1 X_1 \\ \vdots \\ z_r X_r \end{bmatrix} dx_1 \ldots dx_r$$

$$= \xi^{-s}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{[\beta/\alpha]}\frac{(-\beta)_{ka}}{k!}A_{\beta,k}\eta^k\xi^{-hk}\; A^{\mu,\lambda+r+m:\mu_1,\lambda_1;\ldots;\mu_r,\lambda_r}_{\upsilon+r+q,C+p+1:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}$$

$$\left[\begin{matrix} [1-\rho_j/\sigma_j;\xi'_j/\sigma_j,\ldots,\xi_j^{(r)}/\sigma_j]_{1,r},[1-g_j-(S+hk)\gamma_j;N_1\gamma_j,\ldots,N_r\gamma_j]_{1,q} \\ [1-S+\sigma:N_1-n_1,\ldots,N_r-n_r],[1-e_j-(S+hk)\varepsilon_j;N_1\varepsilon_j,\ldots,N_r\varepsilon_j]_{1,p}, \end{matrix}\right.$$

$$\left.\begin{matrix} (a_j;A'_j,\ldots,A_j^{(r)})_{1,c};(\tau'_j,C'_j)_{1,c_1};\ldots;(\tau_j^{(r)},C_j^{r})_{1,c_r} \\ (b_j;B'_j,\ldots,B_j^{(r)})_{1,c};(d'_j,D'_j)_{1,c_1};\ldots;(d_j^{(r)},D_j^{r})_{1,c_r} \end{matrix}\middle| \begin{matrix} Z_1 \\ \vdots \\ Z_r \end{matrix}\right], \qquad \ldots(2.1)$$

where

$$X_i = x_1^{\xi_1^{(i)}}\ldots x_r^{\xi_r^{(i)}}\left(k_1 x_1^{\rho_1}+\ldots+k_r x_r^{\rho_r}\right)^{n_i}, \qquad \ldots(2.2)$$

$$S = \sigma + \sigma_1/\rho_1 + \ldots + \sigma_r/\rho_r, \qquad \ldots(2.3)$$

$$\Psi(k_1,\ldots,k_r) = (\sigma_1\ldots\sigma_r)^{-1}k_1^{-\sigma_1/\rho_1}\ldots k_r^{-\sigma_r/\rho_r} \qquad \ldots(2.4)$$

$$N_i = n_i + \xi_1^{(i)}/\rho_1 + \ldots + \xi_r^{(i)}/\rho_r \qquad \ldots(2.5)$$

and

$$Z_i = z_i\xi_i^{-N_i}k_1^{-\xi_1^{(i)}/\rho_1}\ldots k_r^{-\xi_r^{(i)}/\rho_r}. \qquad \ldots(2.6)$$

The above integral formula (2.1) is valid under the following sufficient conditions:

(a) $k_i > 0,\quad \rho_i > 0,\ n_i \ge 0,\ \xi_j^{(i)} > 0, \forall i,j \in \{1,\ldots,r\},$ ...(2.7)

(b) $\mathrm{Re}(\sigma_i) > 0, i = 1,\ldots,r$ and

$$\mathrm{Re}(S) > -\sum_{i=1}^{r}N_i\delta_i - \min_{1\le i\le m}\{\mathrm{Re}(g_j/\gamma_j)\},$$

where $\delta_i = \min\{\mathrm{Re}(d_j^{(i)}/D_j^{(i)})\}, j = 1,\ldots,\mu_i,$ ...(2.8)

(c) m, p, q, are integers such that $1 \le m \le q$ and $p \ge 0, \varepsilon_j > 0$

$$(j=1,\ldots,p),\ \gamma_j>0(j=1,\ldots,q)\quad \Omega_1 \equiv \sum_{j=1}^{p}\varepsilon_j-\sum_{j=1}^{q}\gamma_j<0,$$

$$\Omega_2 \equiv \sum_{j=1}^{m}\gamma_j-\sum_{j=m+1}^{q}\gamma_j-\sum_{j=1}^{p}\varepsilon_j>0 \text{ and } |\arg(\xi)|<1/2\,\Omega_2\pi, \qquad \ldots(2.9)$$

(d) $A_{\beta,k}$ are arbitrary constants, real or complex and $\beta,k\geq 0$.

(e) Conditions corresponding appropriately (1.4) through (1.6) are satisfied by each of the multivariable A-function occurring in (2.1). Here $H_{p,q}^{m,0}[z]$ denotes the familiar H-function of Fox ([3],p. 408, see also [5], p. 310).

(ii) $$\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1-1}\cdots x_r^{\sigma_r-1}\left(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r}\right)^{\rho} S_\alpha^\beta\left[\eta\left(k_1x_1^{\rho_1}+\cdots+k_rx_r^{\rho_r}\right)^{h}\right]$$

$$A_{\iota,C:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}^{\mu,\lambda:\mu_1;\ldots;\mu_r,\lambda_r}\begin{bmatrix} z_1x_1^{\xi_1}\\ \vdots\\ z_rx_r^{\xi_r}\end{bmatrix}dx_1\ldots dx_r$$

$$=\frac{(-1)^w\gamma^{-s}}{(w)!}\psi(k_1,\ldots,k_r)\sum_{k=0}^{[\beta/\alpha]}\frac{(-\beta)_{k\alpha}}{k!}A_{\beta,k}\eta^k\gamma^{-hk}$$

$$A_{\upsilon+2,C+1:\upsilon_1+1,c_1,\ldots,\upsilon_r+1,c_r}^{\mu,\lambda+2:\mu_1,\lambda_1+1;\ldots;\mu_r,\lambda_r+1}\left[[1-S-hk;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],\right.$$

$$[1-S-hk+u;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],\left(a_j;A'_j;\ldots;A_j^{(r)}\right)_{1,\upsilon};(1-\sigma_1/\rho_1;\xi_1/\rho_1),$$
$$[1-S-hk+u+w;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],\left(b_j;B'_j,\ldots,B_j^{(r)}\right)_{1,C};$$

$$\left.\begin{matrix}\left(\tau_j,C_j\right)_{1,\upsilon_1};\ldots;(1-\sigma_r/\rho_r;\xi_r/\rho_r);\left(\tau_j^{(r)},C_j^{(r)}\right)_{1,\upsilon_r}\\ \left(d'_j,D'_j\right)_{1,c1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,c_r}\end{matrix}\ \middle|\ \begin{matrix}\zeta_1\\ \vdots\\ \zeta_r\end{matrix}\right], \qquad \ldots(2.10)$$

where $L_w^{(u)}(z)$ be the Laguerre polynomials of order u and degree w in $z, w\geq 0, k_i>0, \rho_i>0, \xi_i>0, \mathrm{Re}(\sigma_i)>0, \forall_i=1,\ldots,r$

$$\operatorname{Re}(S) > -\sum_{i=1}^{r}\left(\frac{\xi_i\delta_i}{\rho_i}\right), \operatorname{Re}(\gamma) > 0\,, \qquad \text{...(2.11)}$$

$\psi(k_1,\ldots,k_r), S$ and δ_i being given by (2.4), (2.3) and (2.8), respectively, $\zeta_i = z_i(\gamma k_i)^{-\xi_i/\rho_i}$, $i = 1,..,r$ and conditions given by (1.4) through (1.6) are assumed to hold for the multivariable A-function.

3. Proofs. To prove the main results, we take some assumptions for convenience $\sum n_i s_i$ and $\sum \xi_j^{(i)} s_i$ denote the r-terms sums $\sum_{i=1}^{r} n_i s_i$ and $\sum_{i=1}^{r} \xi_j^{(i)} s_i$ respectively $\forall j = 1,\ldots,r$. ...(3.1)

Also, let

$$\Delta = \int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1 - 1} \ldots x_r^{\sigma_r - 1} f\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right) A^{\mu,\lambda:\mu_1;\ldots;\mu_r,\lambda_r}_{\upsilon,C:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}\begin{bmatrix} z_1 x_1^{\xi_1} \\ \vdots \\ z_r x_r^{\xi_r}\end{bmatrix} dx_1 \ldots dx_r, \qquad \text{...(3.2)}$$

where the X_i are defined by (2.2) and the fuction f is such that the multiple integral converges. On replacing the multivariable A-function occurring in (3.2) by contour integral given by (1.1), under the various conditions stated with (2.1), we find that

$$\Delta = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\ldots\theta_r(s_r)\Phi(s_1,\ldots s_r) z_1^{s_1}\cdots z_r^{s_r}$$

$$\left\{\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1 + \sum \xi_1^{(i)} s_i - 1} \ldots x_r^{\sigma_r + \sum \xi_r^{(i)} s_i - 1}\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)^{\sum n_i s_i}\right.$$

$$\left. f\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right) dx_1 \ldots dx_r\right\} ds_1 \ldots ds_r. \qquad \text{...(3.3)}$$

Now we interrupt the innermost $(x_1,\ldots x_r)$-integral by using the following from of a known result [1,p.173].

$$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1 - 1} \ldots x_r^{\sigma_r - 1}\left(k_1 x_1^{\rho_1} + \ldots k_r x_r^{\rho_r}\right)^{\sigma} f\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right) dx_1 \ldots dx_r = \Psi(k_1,\ldots,k_r)$$

$$\frac{\Gamma(\sigma^*{}_1/\rho_1)\ldots\Gamma(\sigma^*{}_r/\rho_r)}{\Gamma(\sigma^*{}_1/\rho_1 + \ldots + \sigma^*{}_r/\rho_r)}\int_0^\infty z^{\sigma_1/\rho_1 + \ldots + \sigma_r/\rho_r + \sigma - 1} f(z)\, dz, \qquad \text{...(3.4)}$$

where $\Psi(k_1,\ldots,k_r)$ is given by (2.4) and $\min_{1 \le i \le r}\{k_i\rho_i \operatorname{Re}(\sigma_i)\} > 0$ then (3.3) reduces in

the following form

$$\Delta = \frac{\Psi(k_1,\ldots,k_r)}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\ldots\theta_r(s_r)\Phi(s_1,\ldots s_r)Y_1^{s_1}\cdots Y_r^{s_r}$$

$$\frac{\Gamma(\sigma^*_1/\rho_1)\ldots\Gamma(\sigma^*_r/\rho_r)}{\Gamma(\sigma^*_1/\rho_1+\ldots+\sigma_r/\rho_r)}\left\{\int_0^\infty z^{s-\sigma+\sum n_i s_i-1}f(z)dz\right\}ds_1\ldots ds_r, \quad \ldots(3.5)$$

where $\Psi(k_1,\ldots,k_r)$, N_i and S are given by (2.4), (2.5) and (2.3) respectively, and

$$Y_i = z_i k_j^{\sum \xi_j^{(i)}/\rho_j}, \quad \ldots(3.6)$$

$$\sigma_i^\circ = \sigma_j + \sum_{i=1}^{r}\xi_j^{(i)}s_j, \quad \forall j = 1,\ldots,r. \quad \ldots(3.7)$$

Now in the integral (3.5), we set

$$f(z) = z^\sigma H_{p,q}^{m,0}\left[z\xi\left|\begin{matrix}(e_1,\varepsilon_1),\ldots,(e_p,\varepsilon_p)\\(g_1,\gamma_1),\ldots,(g_q,\gamma_q)\end{matrix}\right.\right]S_\beta^\alpha\left[z^h\gamma\right], \quad \ldots(3.8)$$

and evaluate the z-integral by following familiar formula (when n=0), expressing the Mellin transform of Fox's H-function [5,p.311,eq.(3.3)]

$$M\left\{H_{p,q}^{m,n}(zx):s\right\} = \frac{\prod_{j=1}^{m}\Gamma(\beta_j+B_js)\prod_{j=1}^{n}\Gamma(1-\alpha_j-A_js)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j-B_js)\prod_{j=n+1}^{p}\Gamma(\alpha_j+A_js)}z^{-s} \quad \ldots(3.9)$$

Interpret the resulting $(s_1,\ldots,s_r)$–integral as an A-function of r-variables, we will obtain the required result given in (2.1).

Moreover to establish the other main integral (2.10), we can find relationship (3.5) in similar way and then we set

$$f(z) = z^\sigma \exp(-yz)L_w^{(u)}(yz)S_\beta^\alpha\left[\eta z^h\right]. \quad \ldots(3.10)$$

Ecaluate the innermost z-integral by using to a slightly modified version of following well-known integral [2,p.292,eq.(1)]

$$M\left\{e^{-\gamma x}L_m^{(\alpha)}(\gamma x);s\right\} = \frac{\Gamma(\alpha-s+m+1)\Gamma(s)}{m!\Gamma(\alpha-s+1)}\gamma^{-s} \quad \ldots(3.11)$$

If we interpret the resulting multiple contour integral as an A-function of r-variables, we will get desired results (2.10).

4. Special Cases.

(1) For the general class of polynomials, we take the case of Hermite polynomials ([8,p.106,eq.(5.54)] and [7,p.158]) by setting $S_\beta^2[z] = z^{\beta/2} H_\beta\left[\frac{1}{2\sqrt{z}}\right]$ in which case $\alpha = 2$, $A_{\beta,k} = (-1)^k$.

(i) **Integral 1 (a):** The result (2.1) reduces in following form

$$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1 - 1} \ldots x_r^{\sigma_r - 1} \left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)^{\sigma + \beta h/2} \eta^{\beta/2}$$

$$.H_\beta\left[\frac{1}{2\sqrt{\eta\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)^h}}\right] .H_{p,q}^{m,o}\left[\xi\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right) \middle| \begin{matrix} (e_1, \varepsilon_1), \ldots, (e_p, \varepsilon_p) \\ (g_1, \gamma_1), \ldots, (g_q, \gamma_q) \end{matrix}\right].$$

$$A_{\upsilon, C:\upsilon_1, c_1; \ldots; \upsilon_r, c_r}^{\mu, \lambda:\mu_1, \ldots, \mu_r, \lambda_r}\begin{bmatrix} z_1 X_1 \\ \vdots \\ z_r X_r \end{bmatrix} dx_1 \ldots dx_r = \xi^{-s} \Psi(k_1, \ldots, k_r) \sum_{k=0}^{[\beta/2]} \frac{(\beta)! k\alpha}{(\beta - 2k)! k!} \eta^k \xi^{-hk}$$

$$A_{\upsilon+r+q, C+p+1:\upsilon_1, c_1; \ldots; \upsilon_r, c_r}^{\mu, \lambda+r+m:\mu_1, \lambda_1; \ldots; \mu_r, \lambda_r}\left[\begin{matrix} [1 - \rho_j/\sigma_j : \xi'_j/\sigma_j, \ldots, \xi_j^{(r)}/\sigma_j]_{1,r} \\ [1 - S + \sigma : N_1 - n_1, \ldots, N_r - n_r], \end{matrix}\right.$$

$$\begin{matrix} [1 - g_j - (S + hk)\gamma_j ; N_1 \gamma_j, \ldots, N_r \gamma_j]_{1,q}, \\ [1 - e_j - (S + hk)\varepsilon_j ; N_1 \varepsilon_j, \ldots, N_r \varepsilon_j]_{1,p}, \end{matrix}$$

$$\left.\begin{matrix} \left(a_j; A'_j; \ldots; A_j^{(r)}\right)_{1,\upsilon} ; \left(\tau_j, C_j\right)_{1,\upsilon_1} ; \ldots ; \left(\tau_j^{(r)}, C_j^{(r)}\right)_{1,\upsilon_r} \\ \left(b_j; B_j; \ldots; B_j^{(r)}\right)_{1,C} ; \left(d_j, D_j\right)_{1,c_1} ; \ldots ; \left(d_j^{(r)}, D_j^{(r)}\right)_{1,c_r} \end{matrix} \middle| \begin{matrix} Z_1 \\ \vdots \\ Z_r \end{matrix}\right], \qquad \ldots(4.1)$$

valid under the same conditions as obtainable from (2.1).

(ii) **Integral 1(b).** The result (2.10) reduces in following form

$$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1 - 1} \ldots x_r^{\sigma_r - 1} \left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)^{\sigma + \beta h/2}$$

$$.\exp[-\gamma\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)] L_w^{(u)}[\gamma\left(k_1 x_1^{\rho_1} + \ldots + k_r x_r^{\rho_r}\right)] \eta^{\beta/2}$$

$$.H_\beta\left[\frac{1}{2\sqrt{\eta\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^h}}\right]A^{\mu,\lambda:\mu_1,\lambda_1,\ldots,\mu_r,\lambda_r}_{\upsilon,C:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}\left[\begin{matrix}z_1X_1^{\xi_1}\\ \vdots\\ z_rX_r^{\xi_r}\end{matrix}\right]dx_1\ldots dx_r$$

$$=\frac{\left(-1^w\gamma^{-s}\right)}{(w)!}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{[\beta/\alpha]}\frac{(\beta)!(-1)^k}{(k)!(\beta-2k)!}\eta^k\gamma^{-hk}$$

$$.A^{\mu,\lambda+2:\mu_1,\lambda_1+1;\ldots;\mu_r,\lambda_r+1}_{\upsilon+2,C+1:\upsilon_1+1,c_1;\ldots;\upsilon_r+1,c_r}\left[\begin{matrix}[1-S-hk;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],\\ \\ [1-S-hk+u;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],\left(a_j;A_j;\ldots;A_j^{(r)}\right)_{1,\upsilon};(1-\sigma_1/\rho_1;\xi_1/\rho_1),\\ [1-S-hk+u+w;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],(b_j;B'_j;\ldots;B_j^{(r)})_{1,C};\end{matrix}\right.$$

$$\left.\begin{matrix}(\tau'_j,C'_j)_{1,\upsilon_1};\ldots;(1-\sigma_r/\rho_r;\xi_r/\rho_r);\left(\tau_j^{(r)},C_j^{(r)}\right)_{1,\upsilon_r}\\ (d_j,D_j)_{1,c_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,c_r}\end{matrix}\right|\begin{matrix}\zeta_1\\ \vdots\\ \zeta_r\end{matrix}\right], \qquad \ldots(4.2)$$

valid under the same conditions as obtainable from (2.10).

(2) If we set $\alpha=1$ and $A_{\beta,k}=\binom{\beta+V}{\beta}\frac{1}{(V+1)_k}$, the general class of polynomials reduces in Laguerre polynomials ([8,p.106,eq.(15,16)] and [7,p.159]) where Laguerre polynomials are given by

$$L_\beta^{(\upsilon)}[z]=\sum_{k=0}^{\beta}\binom{\beta+\upsilon}{\beta-k}\frac{(-z)^k}{(k)!}\ .$$

(i) Integral 2(a) . The result (2.1) reduces in following form

$$\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1-1}\ldots x_r^{\sigma_r-1}\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^\sigma$$

$$L_\beta^{(\upsilon)}\left[\eta\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^h\right]H^{m,o}_{p,q}\left[\xi\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)\middle|\begin{matrix}(e_1,\varepsilon_1),\ldots,(e_p,\varepsilon_p)\\ (g_1,\gamma_1),\ldots,(g_q,\gamma_q)\end{matrix}\right]$$

$$.A^{\mu,\lambda:\mu_1,\lambda_1;\ldots;\mu_r,\lambda_r}_{\upsilon,C:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}\begin{bmatrix} z_1X_1 \\ \vdots \\ z_rX_r \end{bmatrix}dx_1\ldots dx_r$$

$$=\xi^{-S}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{\beta}\binom{\beta+\upsilon}{\beta-k}\frac{(-\eta)^k}{(k)!}\xi^{-hk}\quad A^{\mu,\lambda+r+m:\mu_1,\lambda_1,\ldots,\mu_r,\lambda_r}_{\upsilon+r+q,C+p+1:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}$$

$$\left[\begin{matrix}[1-\rho_j/\sigma_j:\xi'_j/\sigma_j,\ldots,\xi_j^{(r)}/\sigma_j]_{1,r},[1-g_j-(S+hk)\gamma_j;N_1\gamma_j,\ldots,N_r\gamma_j]_{1,q} \\ [1-S+\sigma:N_1-n_1,\ldots,N_r-n_r],[1-e_j-(S+hk)\varepsilon_j,\ldots,N_1\varepsilon_j,\ldots,N_r\varepsilon_j]_{1,p},\end{matrix}\right.$$

$$\left.\begin{matrix}\left(a_j;A'_j;\ldots;A_j^{(r)}\right)_{1,\upsilon};\left(\tau'_j,C'_j\right)_{1,\upsilon_1};\ldots;\left(\tau_j^{(r)},C_j^{(r)}\right)_{1,\upsilon_r} \\ \left(b_j;B'_j;\ldots;B_j^{(r)}\right)_{1,C};\left(d'_j,D'_j\right)_{1,c_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,c_r}\end{matrix}\middle|\begin{matrix}Z_1\\ \vdots \\ Z_r\end{matrix}\right], \qquad \ldots(4.3)$$

valid under the same conditions as required for (2.1).

(ii) **Integral 2(b).** The result (2.10) reduces in following form

$$\int_0^\infty\cdots\int_0^\infty x_1^{\sigma_1-1}\ldots x_r^{\sigma_r-1}\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^\sigma$$

$$.\exp[-\gamma\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)]L_w^{(u)}[\gamma\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)]\; L_\beta^{(\upsilon)}[\eta\left(k_1x_1^{\rho_1}+\ldots+k_rx_r^{\rho_r}\right)^h]$$

$$A^{\mu,\lambda:\mu_1,\lambda_1;\ldots;\mu_r,\lambda_r}_{\upsilon,C:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}\begin{bmatrix} z_1X_1^{\xi_1} \\ \vdots \\ z_rX_r^{\xi_r} \end{bmatrix}dx_1\ldots dx_r$$

$$=\frac{\left(-1^w\gamma^{-s}\right)}{(w)!}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{\beta}\binom{\beta+\upsilon}{\beta-k}\frac{(-\eta)^k}{(k)!}\gamma^{-hk}$$

$$.A^{\mu,\lambda+2:\mu_1,\lambda_1+1;\ldots;\mu_r,\lambda_r+1}_{\upsilon+2,C+1:\upsilon_1+1,c_1;\ldots;\upsilon_r+1,c_r}\Bigg[[1-S-hk;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],$$

$$[1-S-hk+u;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],\left(a_j;A'_j;\ldots;A_j^{(r)}\right)_{1,\upsilon};(1-\sigma_1/\rho_1;\xi_1/\rho_1),$$

$$[1-S-hk+u+w;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],(b_j;B'_j;\ldots;B_j^{(r)})_{1,C};$$

$$\left(\tau'_j, C'_j\right)_{1,\upsilon_1};\ldots;\left(1-\sigma_r/\rho_r;\xi_r/\rho_r\right);\left(\tau_j^{(r)}, C_j^{(r)}\right)_{1,\upsilon_r} \atop \left(d'_j, D'_j\right)_{1,c_1};\ldots;\left(d_j^{(r)}, D_j^{(r)}\right)_{1,c_r} \left|\begin{matrix}\zeta_1\\ \vdots \\ \zeta_r\end{matrix}\right], \quad \ldots(4.4)$$

valid under the same conditions as obtainable from (2.10).

(3) For the Jacobi polynomials ([8,p. 68, eq. (15,16)] and [7,p. 159]) by setting $S_\beta^1[z] = P_\beta^{(s,t)}[1-2z]$ in which case $\alpha = 1$ and

$$A_{\beta,k} = \binom{\beta+s}{\beta}\frac{(s+t+\beta+1)_k}{(s+1)_k}.$$

(i) Integral 3(a). The result (2.1) reduces in following form

$$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1-1}\ldots x_r^{\sigma_r-1}\left(k_1 x_1^{\rho_1}+\ldots+k_r x_r^{\rho_r}\right)^\sigma$$

$$.P_\beta^{(s,t)}[1-2\eta\left(k_1 x_1^{\rho_1}+\ldots+k_r x_r^{\rho_r}\right)^h]H_{p,q}^{m,0}\left[\xi\left(k_1 x_1^{\rho_1}+\ldots+k_r x_r^{\rho_r}\right)\right.$$

$$\left|\begin{matrix}(e_1,\varepsilon_1),\ldots,(e_p,\varepsilon_p)\\(g_1,\gamma_1),\ldots,(g_p,\gamma_q)\end{matrix}\right] .A_{\upsilon,C:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}^{\mu,\lambda:\mu_1,\lambda_1,\ldots,\mu_r,\lambda_r}\begin{bmatrix}z_1X_1\\ \vdots\\ z_rX_r\end{bmatrix}dx_1\ldots dx_r$$

$$= \xi^{-S}\Psi(k_1,\ldots,k_r)\sum_{k=0}^{\beta}\binom{\beta+s}{\beta-k}\binom{\beta+t+k+s}{k}\xi^{-hk}\; A_{\upsilon+r+q,C+p+1:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}^{\mu,\lambda+r+m:\mu_1,\lambda_1,\ldots,\mu_r,\lambda_r}$$

$$\left[\begin{matrix}[1-\rho_j/\sigma_j:\xi'_j/\sigma_j,\ldots,\xi_j^{(r)}/\sigma_j]_{1,r},[1-g_j-(S+hk)\gamma_j;N_1\gamma_j,\ldots,N_r\gamma_j]_{1,q}\\ [1-S+\sigma:N_1-n_1,\ldots,N_r-n_r],[1-e_j-(S+hk)\varepsilon_j,\ldots,N_1\varepsilon_j,\ldots,N_r\varepsilon_j]_{1,p},\end{matrix}\right.$$

$$\left.\begin{matrix}\left(a_j;A'_j;\ldots;A_j^{(r)}\right)_{1,\upsilon};\left(\tau'_j,C'_j\right)_{1,\upsilon_1};\ldots;\left(\tau_j^{(r)},C_j^{(r)}\right)_{1,\upsilon_r}\\ \left(b_j;B'_j;\ldots;B_j^{(r)}\right)_{1,C};\left(d'_j,D'_j\right)_{1,c_1};\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,c_r}\end{matrix}\left|\begin{matrix}Z_1\\ \vdots\\ Z_r\end{matrix}\right.\right], \quad \ldots(4.5)$$

valid under the conditions as required sufficienty for (2.1).

(ii) Integral 3(b). The relult (2.10) reduces in following form

$$\int_0^\infty \cdots \int_0^\infty x_1^{\sigma_1-1}\ldots x_r^{\sigma_r-1}\left(k_1 x_1^{\rho_1}+\ldots+k_r x_r^{\rho_r}\right)^\sigma$$

$$.\exp[-\gamma(k_1 x_1^{\rho_1}+\ldots+k_r x_r^{\rho_r})]L_w^{(u)}[\gamma(k_1 x_1^{\rho_1}+\ldots+k_r x_r^{\rho_r})]$$

$$P_\beta^{(s,t)}\left[1-2\eta\left(k_1 x_1^{\rho_1}+\ldots+k_r x_r^{\rho_r}\right)^h\right]A^{\mu,\lambda:\mu_1,\lambda_1,\ldots,\mu_r,\lambda_r}_{\upsilon,C:\upsilon_1,c_1;\ldots;\upsilon_r,c_r}\begin{bmatrix} z_1 X_1^{\xi_1} \\ \vdots \\ z_r X_r^{\xi_r}\end{bmatrix}dx_1\ldots dx_r$$

$$=\frac{\left(-1^w\gamma^{-s}\right)}{(w)!}\Psi(k_1,\ldots k_r)\sum_{k=0}^{\beta}\binom{\beta+s}{\beta-k}\binom{\beta+t+k+s}{k}\gamma^{-hk}$$

$$.A^{\mu,\lambda+2:\mu_1,\lambda_1+1;\ldots;\mu_r,\lambda_r+1}_{\upsilon+2,C+1:\upsilon_1+1,c_1;\ldots;\upsilon_r+1,c_r}\Bigg[[1-S-hk;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],$$

$$[1-S-hk+u;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],\left(a_j;A'_j;\ldots;A_j^{(r)}\right)_{1,\upsilon};(1-\sigma_1/\rho_1;\xi_1/\rho_1),$$
$$[1-S-hk+u+w;\xi_1/\rho_1;\ldots;\xi_r/\rho_r],(b_j;B'_j;\ldots;B_j^{(r)})_{1,C};$$

$$\left.\begin{matrix}(\tau'_j,C'_j)_{1,\upsilon_1};\ldots;(1-\sigma_r/\rho_r;\xi_r/\rho_r);\left(\tau_j^{(r)},C_j^{(r)}\right)_{1,\upsilon_r} \\ \left(d'_j,D'_j\right)1,c_1;\ldots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,c_r}\end{matrix}\right|\begin{matrix}\zeta_1\\ \vdots \\ \zeta_r\end{matrix}\Bigg], \qquad \ldots(4.6)$$

valid under the conditions as required sufficiently for (2.10).

(4) If we take $\beta\to 0$ and $\mu=0$ in results (2.1), we obtain a known result obtained by Srivastava and Panda [6,p. 354, eq. (1.8)].

(5) On putting $\beta\to 0$ and $\mu=0$ in result (2.10), we arrive at a known result obtained by Srivastava and Panda [6,p.354,eq.(1.14)].

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(Dedicated to Honour Professor S.P. Singh on his 70th Birthday)

ON UNIFRORM $T\,(C,1)$ SUMMABILITY OF LEGENDRE SERIES

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ABSTRACT

The present paper deals with a theorem on uniform $T(C,1)$ summability of Legendre series under very general condition. It generalizes a very recently known result due to Tripathi and Yadav (2007) on uniform $(N,p,q)(C,1)$ summability of Legendre series under similar condition. It may be noted that the (N,p,q) summability is a particular case of the T-summability method.

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Keywords : Uniform $T(C,1)$ summability, Legendre Series.

1. Introduction. An infinite series with $\Sigma u_n(x)$ with the sequence $\{S_n\}$ of its partial sums is said to be summable $(C,1)$ to a fixed and finite sum $S(x)$ at a point x in an interval E, if the sequence-to-sequence transformation

$$C_n^1(x) = \frac{1}{n}\sum_{m=1}^{n} S_m(x) \tag{1.1}$$

tends to $S(x)$ as $n \to \infty$. [Titchmarsh (1959), p. 411].

Let $T = \left[a_{n,k}\right]$ be an infinite triangular matrix with entries $a_{n,k}$ over reals or complexes and

$$a_{n,k,} = 0 \text{ for } k > n. \tag{1.2}$$

The sequence-to-sequence transformation

$$T_n(x) = \sum_{k=1}^{n} a_{n,k} C_k^1(x) \tag{1.3}$$

definies the n$^{\text{th}}$ T-mean of the sequence $\left\{C_n^1(x)\right\}$ of the $(C,1)$ means of the sequence $\{S_n(x)\}$ of partial sums of the series $\Sigma u_n(x)$, or the n^{th} $T(C,1)$ mean of the series $\sum u_n(x)$ at the point x in the interval E.

If there exists a function $S(x)$ such that

$$T_n(x) - S(x) = O(1) \tag{1.4}$$

as $n \to \infty$, unformly in E, the series $\Sigma u_n(x)$ is said to be summable $T(C,1)$ unformly in E to the sum $S(x)$.

The Legendre series associated with a Lebesgue integrable function $f(x)$ in the range (-1,1) is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) \tag{1.5}$$

where

$$a_n = \left(n + \frac{1}{2}\right) \int_{-1}^{1} f(x) P_n(x) dx \tag{1.6}$$

and the n^{th} Legendre polynomial $P_n(x)$ is defined by the generating function

$$\frac{1}{\sqrt{1 - 2xz + z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n . \tag{1.7}$$

We use the following notations

$$\psi(t) = \psi_0(t) = f\{\cos(\theta - t)\} - f(\cos\theta)$$

and

$$N_n(t) = \sum_{k=1}^{n} a_{n,k} \frac{\sin(k+1)/2 \sin kt/2}{k \sin^2 t/2} .$$

2. Main Result. In this section, our aim is to study uniform $T(C,1)$ summability of Legendre series (1.5) under very general conditons by establishing the following :

Theorem. Let $T = (a_{n,k})$ be an infinite regular triangular matrix with entries $(a_{n,k})$ as a sequence of non-negative reals, non-decreasing with respect to k and

$$A_{n,m} = \sum_{k=1}^{m} a_{n,k} . \tag{2.1}$$

Let $\{p_n\}$ be a non-negative, monotonic non-increasing sequence of real coefficients such that its n^{th} partial sum $P_n \to \infty$ as, $n \to \infty$. Let $\lambda(t)$ and $\mu(t)$ be two positive functions of t such that $\lambda(t)$, $\mu(t)$ and $t\lambda(t)/\mu(t)$ increase monotonically with t and

$$\lambda(n) P_n = O[\mu(P_n)], \text{ as } n \to \infty \tag{2.2}$$

If

$$\int_0^t |\psi(u)| du = O\left[\frac{\lambda(1/t)p_\tau}{\mu(P_\tau)}\right], \text{ as } t \to 0 \tag{2.3}$$

and

$$\int_t^\eta \frac{|\psi(u)|}{u^2} A_{n,[1/u]} du = o(1), \tag{2.4}$$

as $t \to 0$, unformly in a set E defined in the interval (-1,1), where

$\eta = \min[arc\cos u - arc\cos(u+\alpha)]$

for u in (-1,1-α), $\alpha>1$; then the Legendre series (1.5) is summable $T(C,1)$ unformly in E to the sum $f(x)$.

3. Lemmas. The following lemmas are needed in order to prove our main theorem:

Lemma1. [Lal (2000)] : If $(a_{n,k})$ is non-negative and non-decreasing with $k \le n$ then for $0 \le a < b \le \infty$, $0 \le t \le \pi$ and any n $\left|\sum_{k=a}^{b} a_{n,n-k} e^{i(n-k)t}\right| = O(A_{n,\tau})$.

Lemma 2. If we write

$$N_n(t) = \sum_{k=1}^{n} a_{n,k} \frac{\sin(k+1)t/2 \sin kt/2}{k \sin^2 kt/2}$$

then

$$N_n(t) = \begin{cases} O(n) & \text{for } 0 \le t \le 1/n \\ O\left(\frac{1+A_{n,\tau}}{nt^2}\right) & \text{for } 1/n \le t \le \eta. \end{cases}$$

Proof of Lemma 2. For $0 \le t \le 1/n$,

$$|N_n(t)| \le \sum_{k=1}^{n} |a_{n,k}| \frac{|\sin(k+1)t/2 \sin kt/2|}{k|\sin^2 t/2|}$$

$$= O\left(\sum_{k=1}^{n} k|a_{n,k}|\right)$$

$$= O\left[(n)\sum_{k=1}^{n} |a_{n,k}|\right]$$

$$= O(n), \text{ as } n \to \infty.$$

Using regularity conditions for T-method of summation for $1/n \le t \le \eta$

$$|N_n(t)| = \left|\sum_{k=1}^{n} a_{n,k} \frac{\sin(k+1)t/2 \sin kt/2}{k\sin^2 t/2}\right|$$

$$= \left|\sum_{k=1}^{n} a_{n,k} \frac{\{\cos t/2 - \cos(2k+1)t/2\}}{k\sin^2 t/2}\right|$$

$$\le \left|\sum_{k=1}^{n} a_{n,k} \frac{\cos t/2}{k\sin^2 t/2}\right| + \left|\sum_{k=1}^{n} a_{n,k} \frac{\cos(2k+1)t/2}{k\sin^2 t/2}\right|$$

$$= O\left(1/nt^2\right) + O\left[1/nt^2 \left|R\sum_{k=1}^{n} a_{n,k} e^{i(2k+1)t/2}\right|\right]$$

$$= O\left(\frac{1}{nt^2}\right) + O\left(\frac{1}{nt^2}\right) \cdot \left|\mathrm{Re}^{i(t/2)}\right| \left|R\sum_{k=1}^{n} a_{n,k} e^{ikt}\right|$$

$$= O\left(\frac{1}{nt^2}\right) + O\left(\frac{1}{nt^2}\right) \left|R\sum_{k=1}^{n} a_{n,n-k} e^{i(n-k)t}\right|$$

$$= O\left(\frac{1}{nt^2}\right) + O\left(\frac{A_{n,\tau}}{nt^2}\right),$$

as $n \to \infty$.

4. Proof of the theorem. Then n^{th} partial sum $S_n(x)$ of the Legendre series (1.5) at any point x in (-1,1) is given after a well known computation by

$$(4.1) \qquad S_n(x) - f(x) = \frac{1}{\pi\sqrt{\sin\theta}} \int_0^{\pi} \frac{f\{\cos(\theta - t)\} - f(\cos\theta)}{\sin t/2} \sin(n+1)t \sqrt{\sin(\theta - t)}\, dt + o(1)$$

$$= O\left[\int_0^{\pi} [f\{\cos(\theta - t)\} - f(\cos\theta)] \frac{\sin(n+1)t}{\sin t/2} dt\right] + o(1)$$

$$= O\left[\int_0^{\pi} \psi(t) \left[\frac{\sin(n+1)t}{\sin t/2}\right]\right] + o(1)$$

where,

$\eta = \min[arc\cos u - arc\cos(u+\alpha)]$

for u in (-1,1-α), $\alpha>0$.

Now, the $(C,1)$ mean $\sigma_n(x)$ of the Legendre series (1.5) at x in (-1,1) will be given by

$$\sigma_n(x)-f(x)=\frac{1}{n}\sum_{m=0}^{n-1}\{S_m(x)-f(x)\}$$

$$=\frac{1}{n}\int_0^n \frac{\psi(t)}{\sin t/2}\left\{\sum_{m=0}^{n-1}\sin(m+1)t\right\}dt+O(1)$$

$$=\frac{1}{n}\int_0^n \frac{\psi(t)\sin(n+1)t/2\sin nt/2}{\sin^2 t/2}dt+O(1). \tag{4.2}$$

Further, following (1.3), we have $T(C,1)$ mean $T_n(x)$ of the sequence $S_n(x)$ of partial sums of the series (1.5) given by

$$T_n(x)-f(x)=\sum_{k=1}^{n}a_{n,k}\{\sigma_k(x)-f(x)\}$$

$$=\int_0^\eta \psi(t)\left(\sum_{k=1}^{n}a_{n,k}\frac{\sin(k+1)t/2\sin kt/2}{k\sin^2 t/2}\right)dt+O(1)$$

$$=\int_0^\eta \psi(t)N_n\,dt+O(1) \qquad \text{(say)} \tag{4.3}$$

where

$$N_n(t)=\sum_{k=1}^{n}a_{n,k}\frac{\sin(k+1)t/2\sin kt/2}{k\sin^2 t/2}.$$

Now, if we show that

$$T_n(x)-f(x)=o(1) \tag{4.4}$$

as $n\to\infty$ unformly in a set E then the Legendre series (1.5) will be summabile $T(C,1)$ unformly in E to the sum $f(x)$.

Let us write

$$I=T_n(x)-f(x)=\int_0^\eta \psi(t)N_n(t)dt+O(1)$$

$$=\int_0^{1/n}+\int_{1/n}^{\eta}+O(1)$$

$$= I_1 + I_2 + O(1), \text{ say.} \tag{4.5}$$

Firstly, we consider I_1. Now,

$$|I_1| \le \int_0^{1/n} |\psi(t)||N_n(t)|dt$$

$$= O(n)\int_0^{1/n} |\psi(t)|dt \text{, using lemma 2}$$

$$= O(n)o\left[\frac{\lambda(n)p_n}{k(P_n)}\right], \text{ using (2.3)}$$

$$= o\left[\frac{\lambda(n)P_n}{k(P_n)}\right], \text{since } np_n \le P_n \text{ by the conditon on } \{p_n\}.$$

$$= o(1), \text{ using (2.2)} \tag{4.6}$$

as $n \to \infty$, unformly in E.

Next, we consider I_2. Here,

$$|I_2| \le \int_{1/n}^{\eta} |\psi(t)||N_n(t)|dt$$

$$= O\left(\frac{1}{n}\right)\int_{1/n}^{\eta} \frac{|\psi(t)|}{t^2}dt + O\left(\frac{1}{n}\right)\int_{1/n}^{\eta} \frac{|\psi(t)|}{t^2}A_{n,\tau}dt$$

$$= O(1/n).I_{2.1} + O(1/n), I_{2.2}, \text{ say.} \tag{4.7}$$

Now,

$$I_{2.1} = \left[\frac{1}{t^2}.o\left\{\frac{\lambda(1/t)p_\tau}{k(P_\tau)}\right\}\right]_{1/n}^{\eta} + \int_{1/n}^{\eta} o\left\{\frac{\lambda(1/t)p_\tau}{k(P_\tau)}\right\}\frac{1}{t^3}dt$$

$$= O(n^2)O\left[\frac{\lambda(n)p_n}{k(P_n)}\right] + O\left[\frac{\lambda(n)p_n}{k(P_n)}\right]\int_{1/n}^{\eta} t^{-3}dt$$

$$= O\left[\frac{n\lambda(n)P_n}{k(P_n)}\right] + O\left[\frac{\lambda(n)p_n}{k(P_n)}\right].\left[\frac{t^{-2}}{-2}\right]_{1/n}^{\eta}$$

$$= O(n) + O\left[\frac{n^2\lambda(n)P_n}{k(P_n)}\right]$$

$$= O(n) + O(n)$$

$$= O(n)$$

so that

$$O(1/n).I_{2.1} = o(1), \text{ as } n \to \infty, \text{ unformaly in } E. \tag{4.8}$$

Lastly considering $I_{2.2}$, we have

$$I_{2.2} = \int_{1/n}^{\eta} \frac{|\psi(t)|}{t^2} A_{n,\tau} dt$$

$$= o(1), \text{ as } n \to \infty,$$

unformly in E, on using (3.4), so that

$$O(1/n).I_{2.2} = O(1/n).o(1)$$

$$= o(1), \tag{4.9}$$

as $n \to \infty$, unformly in E.

Combining (4.7), (4.8) and (4.9), we get

$$I_2 = o(1), \tag{4.10}$$

as $n \to \infty$, unformly in E.

Lastly, combining (4.5), (4.6) and (4.10), we obtain the required result in (4.4). This completes the proof of our theorem.

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FINITE M/G/1 QUEUEING SYSTEM WITH BERNOULLI FEEDBACK UNDER OPTIMAL *N*-POLICY

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ABSTRACT

This paper ivestigates $M/G/1$ queue with Bernoulli feedback under an optimal control policy. According to N-policy, the server renders service only when N customers are accumulated in the system; once he initiates service, continues to provide it till system becomes empty. The system size under steady state conditions are determined by using generating function and supplementary variable technique. The analytical results for average queue length and mean response time are obtained. The optimal N-value is obtained by minimizing the total operation cost of the system.

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1. Introduction. Feedback queueing networks are important in many applications where the customer getting incomplete service may try to seek service repeatedly till his service gets completed. Such situations arise in manufacturing, data communications, telecommunications, packet transmissions, etc. Several authors have investigated the feedback queueing systems in different frameworks; some of the recent works are as given below. Disney et al. (1984) derived stationary queue length and waiting time distribution in single server feedback queues. Vanden et al. (1991) studied the $M/G/1$ queue with processor sharing and its relation to a feedback queue. Sojourn time in vacation queues and polling systems with Bernoulli feedback was discussed by Takine et al. (1991). Rege (1993) discussed the $M/G/1$ queue with Bernoulli feedback. Adve and Nelson (1994) gave the relationship

between Bernoulli and fixed feedback policies for the $M/G/1$ queue. Thangaraj and Santhakumaran (1994) analysed Sojourn times in queues with a pair of instantaneous independent Bernoulli feedback. Bhattacharya et al. (1995) studied on adoptive optimization in multiclass $M/GI/1$ queues with Bernoulli feedback. Takagi (1996) had studied the response time in $M/G/1$ queues with service in random order and Bernoulli feedback. Fujian and Yang (1998) discussed queue length performance of some non-exhaustive polling models with Bernoulli feedback. Choi, et al. (2003) discussed an $M/G/1$ queue with multiple types of feedback with gated vacations and *FCFS* policy.

Many researchers have incorporated the concept of N-policy in queueing system in which the server turns on whenever N or more customers are present in the system. Tang (1994) obtained takacs type equation in the $M/G/1$ queue with N-policy adn setup times. Piscataway and Choudhury (1997) considered a Poisson queue under N-policy with a general setup time. Lee and Park (1997) discussed an optimal strategy based on N-policy for production system with early setup. Jain (1997) has introduced an optimal N-policy for single server Markovian queue with breakdown, repair and state dependent arrival rate. Artalejo (1998) gave some results for the $M/G/1$ queue under N-policy. Gupta (1999) considered N-policy queueing system with finite source and warm spares. Hur and Paik, (1999) analysed effect of different arrival rates on the N-policy for $M/G/1$ with server setup. Kumar et al. (2002) considered the N-policy $M/G/1$ feedback queue with varying arrival rates. Jain and Singh (2003) have discussed an optimal N-policy for the state-dependent $M/E_k/1$ queue with server breakdowns. Jain (2003) analysed N-policy for redundant repairable system with additional repairmen. Pearn adn Chang (2004) presented optimal management of the N-policy $M/E_k/1$ queueing system with a removable service station. Berman and Larson (2004) have analysed a queu control model for retail services having back room operations and cross-trained workers.

Several authors used the supplementary variable technique in $M/G/1$ queue. Hokstad (1975) has applied supplimentary variable technique in the $M/G/1$ queue. Wang and Ke (2000) employed a recursive method using the supplementary variable technique to established optimal control policy at a minimum cost involved.

The purpose of this paper is to obtain the mean response time and optimal operating N-policy for finite $M/G/1$ queueing model with Bernoulli feedback. The rest of the paper is organized as follows. We provide notations and terminology related to model in section 2. In section 3, the probability generating function technique to obtain steady state probability distribution of the number of units in the system is discussed. The supplementary variable technique by treating the remaining service time as supplementary variable is used. The explicit results for

the average number of customers in the system and the mean response time are determined in section 4. The optimal operating N-policy is also stated to minimize the total expected cost per customer time. In section 5, we deduced some particular cases which match with earlier existing results. Finally conclusion is drawn in seciton 6.

2. The Model Analysis and Description. Consider $M/G/1$ Bernoulli feedback state dependent single server queue with general service time distribution and finite capacity K under N-policy. The customers arrive at the system according to Poisson process with arrival rate λ and the system permits no arrivals when the number of customers in the system is reached to K. The service time of each customer is independently identically distributed (i.i.d.) random variable with distribution function $B(x)(x \geq 0)$. A single server renders the service according to *FCFS* discipline. As soon as the system becomes empty, the server turns off and may not operate until N customers are accumulated in the system. A customer departs from the system after completion of the service with probability σ. If the service is not successfully completed then the customer again joins the system and is cycled back into service with probability (1-σ). We apply supplementary variable technique by introducing the new variable X denoting the remaining service time.

The following notations and terminology are used to formulate the mathematical model :

ρ -traffic intensity such that $0 < \sigma \leq 1$.
σ - Bernoulli feedback parameter.
N - Threshold parameter in turn on policy.
X - remaining service time for the customers being served.
W - expected waiting time of the costomers.
pdf - probability density function.
LST - Laplace Stieltjes Transform.
pgf - probability generating function.
B - service time random variable.
$B(x)$ - service time distribution.
$b(x)$ - *pdf* of service time distribution.
$B^*(\theta)$ - *LST* service time distribution.
$B^{*(i)}(\theta)$- i^{th} order derivative of service time distribution with respect to θ.
H - service time random variable in Bernouli fcedback queue.
$H(x)$ - service time distribution of H.
$h(x)$ - *pdf* of service time distribution of H.
$H^*(\theta)$ - *LST* service time distribution of H.

$H^{*(i)}(\theta)$- i^{th} order derivative of H with respect to θ.

$p_{0,0}(x)$ = Pr (the system being empty at time t)

$p_{0,n}(x)$ = Pr (there are n customers in the system at time t when the server is turned off, where $n=1,2,...,N$-1.).

$p_{1,n}(x)$ = Pr(there are n customers in the system at time t when the server is turned on and working, $n=1,2,...$).

$p^*_{1,n}(\theta)$ = LST of $p_{1,n}(x)$.

$G_0(z)=\sum_{n=0}^{N-1} z^n p_{0,n}$, pgf of $p_{0,n}(x)$.

$G_1(z,0)=\sum_{i=1}^{K} z^n p_{1,n}(0)$,$pgf$ of $p_{1,n}(x)$.

$G_1^*(z,\theta)=\sum_{n=1}^{K} z^n p_{1,n}^*(\theta)$,$pgf$ of $p_{1,n}^*(\theta)$.

$G(z)$ = pgf of the number of customers in the system.

Let us assume that the customer is feedback the end of the queue and will return again and again till the service is completed. The total service time $H(x)$ of the customer is the time from the initialization of service upto final departure from the system. It is noted that (cf. Kumar et al., 2002)

$$H^*(\theta)=\frac{\sigma B^*(\theta)}{\{1-(1-\sigma)B^*(\theta)\}} \quad ...(1)$$

Thus, the mean service time $H^{*(1)}(0)$ and the second moment $H^{*(2)}(0)$ of the service time of customer is given by (cf. Takine et al. 1991)

$$H^{*(1)}(0)=\frac{-B^{*(1)}(0)}{\sigma}, \quad H^{*(2)}(0)=\frac{B^{*(2)}(0)}{\sigma}+\frac{2(1-\sigma)}{\sigma^2}\{B^{*(1)}(0)\}^2 \quad ...(2)$$

3. The Analysis. We now proceed to analyze M/G/1 queueing system with Bernoulli feedback under N-policy by using gererating function and supplementary variable techniques, treating the remaining service time as supplementary variable X to obtain steady state results of the model under investigation. Let us define

$$p_{1,n}(x,t)dx=\Pr.\{Z(t)=n, x<X(t)\le x+dx\} \qquad x\ge 0,\ \ n=1,2...$$

$$P_{1,n}(t)=\int_0^\infty P_{1,n}(x,t)dx, n=1,2,... \quad ...(3.0)$$

The equations that are governing the model are as follows:

$$\frac{d}{dt}p_{0,0}(t) = -\lambda_0 p_{0,0}(t) + p_{1,1}(0,t) \qquad \text{...(3.1)}$$

$$\frac{d}{dt}p_{0,n}(t) = -\lambda_0\beta_0 p_{0,n}(t) + \lambda_0\beta_0 p_{1,1}(0,t), \qquad 1 \le n \le N-1 \qquad \text{...(3.2)}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)p_{1,1}(x,t) = -\lambda p_{1,1}(x,t) + p_{1,2}(0,t)h(x) \qquad \text{...(3.3)}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)p_{1,n}(x,t) = -\lambda p_{1,n}(x,t) + \lambda p(x,t) + p_{1,n+1}(0,t)h(x), \qquad 2 \le n \le N-1 \qquad \text{...(3.4)}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)p_{1,N}(x,t) = -\lambda p_{1,N}(x,t) + \lambda p_{1,N-1}(x,t) + \lambda p_{0,N-1}(x,t)p_{1,N+1}(0,t)h(x) \qquad \text{...(3.5)}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)p_{1,n}(x,t) = -\lambda p_{1,n}(x,t) + \lambda p_{1,n-1}(x,t) + p_{1,n+1}(0,t)h(x), N+1 \le n \le K-1 \qquad \text{...(3.6)}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)p_{1,K}(x,t) = -\lambda p_{1,K-1}(x,t)\ . \qquad \text{...(3.7)}$$

For steady state, we use the following notations

$$p_{0,n} = \lim_{t\to\infty} p_{0,n}(t), \qquad n = 0,1,2,\ldots,N-1$$

$$p_{1,n} = \lim_{t\to\infty} p_{1,n}(t), \qquad n = 1,2,\ldots,K$$

$$p_{1,n}(x) = \lim_{t\to\infty} p_{1,n}(x,t), \qquad n = 1,2,\ldots,K.$$

Further let us define

$$p_{0.N-1}(x) = p_{0.N-1}h(x)\ . \qquad \text{...(4.0)}$$

In steady state, the equations (3.1)–(3.7) reduce to the following:

$$0 = -\lambda_0 p_{0.0} + p_{1.1}(0) \qquad \text{...(4.1)}$$

$$0 = -\lambda_0\beta_0 p_{0.n} + \lambda_0\beta_0 p_{1.1}, \qquad 1 \le n \le N-1 \qquad \text{...(4.2)}$$

$$-\frac{d}{dx}p_{1.1}(x) = \lambda p_{1.1}(x) + p_{1.2}(0,t)h(x) \qquad \text{...(4.3)}$$

$$-\frac{d}{dx}p_{1.n}(x) = \lambda p_{1.n}(x) + \lambda p_{1.n-1}(x) + p_{1.n+1}(0)h(x), \qquad 2 \le n \le N-1 \qquad \text{...(4.4)}$$

$$-\frac{d}{dx}p_{1.N}(x) = -\lambda p_{1.N}(x)+\lambda p_{1.N-1}(x)+\lambda p_{0.N-1}(x)+p_{1.N+1}(0)h(x), \qquad \text{...(4.5)}$$

$$-\frac{d}{dx}p_{1.n}(x) = -\lambda p_{1.n}(x)+\lambda p_{1.n-1}(x)+p_{1.n+1}(0)h(x), \quad N+1 \le n \le K-1 \qquad \text{...(4.6)}$$

$$-\frac{d}{dx}p_{1.K}(x) = \lambda p_{1.n-1}(x)\ . \qquad \text{...(4.7)}$$

Using equations (4.1)–(4.2), we obtain

$$p_{1.1}(0)=\lambda_0 p_{0.0}=\lambda_0 p_{0.n}, \qquad 1\le n\le N-1 \qquad \text{...(5.0)}$$

When we take LST of equations (4.3)-(4.7), we get the following:

$$(\lambda-\theta)p^*_{1.1}(\theta)=p_{1.2}(0)H^*(\theta)-p_{1.1}(0) \qquad \text{...(5.1)}$$

$$(\lambda-\theta)p^*_{1.n}(\theta)=\lambda p^*_{1.n-1}(\theta)+p_{1.n+1}(0)H^*(\theta)-p_{1.n}(0), \quad 2\le n\le N-1 \qquad \text{...(5.2)}$$

$$(\lambda-\theta)p^*_{1.N}(\theta)=\lambda p^*_{1.N-1}(\theta)+p_{1.N+1}(0)H^*(\theta)+\lambda p_{0.N-1}(0)H^*(\theta)-p_{1.N}(0), \qquad \text{...(5.3)}$$

$$(\lambda-\theta)p^*_{1.n}(\theta)=\lambda p^*_{1.n-1}(\theta)+p_{1.n+1}(0)H^*(\theta)-p_{1.n}(0), \qquad N+1\le n\le K-1 \qquad \text{...(5.4)}$$

$$-\theta p^*_{1.K}(\theta)=\lambda p^*_{1.K-1}(\theta)-p_{1.K}(0) \qquad \text{...(5.5)}$$

Using (5.0),we get

$$\lambda_0 p_{1.N-1}H^*(\theta)=p_{1.1}(0)H^*(\theta)\ . \qquad \text{...(6)}$$

Now using this result equation (5.3) reduces to

$$(\lambda-\theta)p^*_{1.N}(\theta)=\lambda p^*_{1.N-1}(\theta)+p_{1.N+1}(0)H^*(\theta)+p_{1.1}(0)H^*(\theta)-p_{1.N}(0) \qquad \text{...(7)}$$

Now, adding equations (5.1)–(5.5), we get

$$\sum_{n=1}^{K}p^*_{1.n}(\theta)=\frac{\lambda(1-\beta)}{\theta}p_{1.K-1}(\theta)+\left\{\frac{1-H^*(\theta)}{\theta}\right\}\sum_{n=1}^{K}p_{1.n}(0)\ . \qquad \text{...(8)}$$

Applying L'Hospital' s rule in (8), we get

$$\sum_{n=1}^{K}p_{1.n}(\theta)=\frac{b_1}{\sigma}\sum_{n=1}^{K}p_{1.n}(0) \qquad \text{...(9)}$$

where $b_1=B^{*(1)}(0)$

3.1 Probability Generating Function Technique. The equations (5.1) and (5.2) are diffcult to be solved recursively. In view of this we proceed to obtain analytic solution for $p_{1,n}, p^*_{1,n}(0)$ $(n=1,2...)$ in closed-form by using probability generating function technique.

Using $p_{0,0} = p_{0,n}(n = 0,1,2,\ldots,N-1)$, we have

$$G_0(z) = \frac{1-z^N}{1-z}p_{0,0} \qquad \ldots(10)$$

Multiplying (5.0) and (5.5) by appropriate power of z and summing, we get the following equation

$$(\lambda-\theta-\lambda z)G_1^{\circ}(z,\theta) = \left\{\frac{H^{\circ}(\theta)}{z}-1\right\}G_1(z,0)+\lambda z^N H^{\circ}(\theta)p_{0,0} - \lambda H^{\circ}(\theta)p_{0,0} + \lambda z^K p_{1,K}^{\circ}(\theta) \ldots(11)$$

Now putting $\theta = \lambda - \lambda z$ in (1.1), we have

$$G_1(z,0) = \frac{\lambda(1-z^N)H^{\circ}(\lambda-\lambda z)}{\{(H^{\circ}(\lambda-\lambda z)/z)-1\}}p_{0,0} + \frac{\lambda z^K p_{1,K}^{\circ}(\lambda-\lambda z)}{\{(H^{\circ}(\lambda-\lambda z)/z)-1\}} \qquad \ldots(12)$$

Substituting the value of $G_1(z,0)$ from (12) into (11), we find

$$G_1^{\circ}(z,0) = \frac{\lambda(1-Z^N)}{(\lambda-\theta-\lambda z)}\left[\left\{\frac{H^{\circ}(\theta)}{z}-1\right\}\left\{\frac{zH^{\circ}(\lambda-\lambda z)}{H^{\circ}(\lambda-\lambda z)-z}\right\}-H^{\circ}(\theta)\right]p_{0,0}+\lambda z^K p_{1,K}^{\circ}(\lambda-\lambda z) \ldots(13)$$

By Putting $\theta = 0$ in (13), we obtain $G_1^{\circ}(z,0)$ as

$$G_1^{\circ}(z,0) = \left[\frac{(1-Z^N)H^{\circ}(\lambda-\lambda z)}{H^{\circ}(\lambda-\lambda z)-z} - \frac{(1-Z^N)}{(1-Z)}\right]p_{0,0} + \lambda z^K p_{1,K}^{\circ}(\lambda-\lambda z) = \sum_{n=1}^{\infty} z^n p_{1,n}^{\circ}(0) \quad (14)$$

Now the probability generating function of the number of customers in the system is given by

$$G(z) = G_0(z) + G_1^{\circ}(z,0) = \left[\frac{(1-z^N)H^{\circ}(\lambda-\lambda z)}{H^{\circ}(\lambda-\lambda z)-z}\right]p_{0,0} + \lambda z^K p_{1,K}^{\circ}(\lambda-\lambda z). \qquad \ldots(15)$$

The probability $p_{0,0}$ can be determined using the normalizing condition

$$Np_{0,0} + \sum_{i=1}^{K} p_{1,i}^{\circ} = 1 \qquad \ldots(16)$$

To determine $p_{1,K}^{\circ}(0)$, we use recursive approach and obtain

$$p_{1,K}^{\circ}(0) = -\left(H^{*(\cdot)}(0)\sum_{i=1}^{K} p_{1,i} + \sum_{i=1}^{K} p_{1,i}^{\circ}\right)$$

which gives $p_{1,K}^{\circ}(0) = \left(\sum_{i=1}^{K-1} p_{1,i}\right)$ ...(17)

4. The Expressions for performance Measures. We now, propose to find the following performance characteristics in exlicit form by using generating functions tecnique as follows:

(i) Expected queue length. The expected number of coustomers in the queue is obtained by using (17) as

$$L = G'(z)\big|_{z=1}$$

$$\left[\frac{N(N-1)\sigma(\sigma-\rho)+2N\rho(\sigma-\rho)+N\lambda^2\sigma^2 H^{*(2)}(0)}{2(\sigma-\rho)^2}\right]p_{0,0}$$

$$+\lambda K(K-1)\,\overset{*}{p}_{1,K}(0)-2\lambda^2 K p_{1,K}^{*(')}(0)+\lambda^3 K p_{1,K}^{*('')}(0) \qquad \text{...(18)}$$

where $p_{1,K}^{*(')}(0)$ and $p_{1,K}^{*('')}(0)$ are determined by using (17)

and $H^{*(2)} = \left[\dfrac{B^{*(2)}(0)}{\sigma}+\dfrac{2(1-\sigma)}{\sigma^2}\left\{B^{*(1)}(0)\right\}^2\right],$

where $B^{*(2)}(0)$ stands for the second moment of the service time.

(ii) Mean response time.

The mean response time is given by

$$E(R) = W + B^{*(1)}(0) = \frac{L}{\lambda} + B^{*(1)}(0) \qquad \text{...(19)}$$

where the value of L is given by (18)

(iii) Optimal N-Policy

Let $E[I], E[\dot{B}]$ and $E[C]$ denote the expected length of the idle period. the busy period and busy cycle, repectively, such that $E[C] = E[I], E[B]$. As the length of the idle period is the sum of N exponetial random variables each having a mean of $1/\lambda$ so that we get

$$E[I] = \frac{N}{\lambda}.$$

The long-run fractions of time of idle and busy server are given as follows :

$$\frac{E[I]}{E[C]} = G_0(1) = Np_{0,0} \qquad \text{...(20)}$$

$$\frac{E[B]}{E[C]} = G_1'(1,0) = \left(\frac{N\rho}{\sigma-\rho}\right)p_{0,0} + \lambda K p_{1,K}(0) - \lambda p_{1,k}^{*(')}(0) \qquad \text{...(21)}$$

The number of busy cycles per unit time is obtained as

$$\frac{1}{E[C]} = \lambda p_{0,0} \qquad \text{...(22)}$$

we now consider a cost function in order to get optimal value of threshold parameter N as given below:

$$E\{F(N)\} = C_h L + C_f \frac{E[I]}{E[C]} + C_0 \frac{E[B]}{E[C]} + (C_s + C_d)\frac{1}{E[C]} \qquad \text{...(23)}$$

where, C_h = Holding cost per unit time per customer present in the system.
C_f = Cost incurred per unit time for keeping the server off.
C_o = Cost incurred per unit time for keeping the server on.
C_s = Start-up cost per unit time for turning the server on.
C_d = Shutdown cost per unit time for turning the server off.

Since $\frac{1}{E[C]}$ is independent of the decision variable N, neglecting fourth term of (23), we get a new cost function per unit time to be minimized as follows:

$$E\{C(N)\} = C_h \left\{\frac{N(N-1)}{2}\right\} + C_f N p_{0,0} + C_0 N\left(\frac{N}{\sigma-\rho}\right) p_{0,0}. \qquad \text{...(24)}$$

To find the optimal value of threshold parameter N (say N^*), we put $\frac{dE\{C(N)\}}{dN} = 0$, yielding to

$$N^* = \frac{1}{C_h}\left[\frac{C_h}{2} - \left\{C_f + \frac{C_0 \rho}{(\sigma-\rho)}\right\} p_{0,0}\right] \qquad \text{...(25)}$$

We observe that N^* is not an integer, the best positive integeral value of N is obtained by rounding off the value of N^*.

5. Particular Cases. In this section , some special cases by taking appropriate parameter, will be obtained as follows

Case1: The $M/G/1$ Model with N-Policy and Bernoulli Feedback. Putting $\lambda_0 = \lambda$ and $k \to \infty$ i.e. the case is of infinite capacity system, we get results as follows:

The average numbers of customers in the system and mean response times are

$$L = \frac{N-1}{2} + \frac{\rho}{\sigma} + \frac{\lambda^2 \sigma}{2(\sigma-\rho)}\left[\frac{B^{*(2)}(0)}{\sigma} + \frac{2(1-\sigma)}{\sigma^2}\left\{B^{*(1)}(0)\right\}^2\right] \qquad \text{...(26)}$$

and $$E(R)=\frac{N-1}{2\lambda}+\frac{2b_1}{\sigma}+\frac{\lambda B^{*(2)}(0)}{2(\sigma-\rho)}+\frac{\lambda(1-\sigma)}{\sigma(\sigma-\rho)}\{B^{*(1)}(0)\}^2 \quad \text{...(27)}$$

By changing the specific distributiobn for service time, the expressions for the average number of customers in the system and mean response time reduces to

$$L=\begin{cases}\dfrac{N-1}{2}+\dfrac{\rho}{\sigma}+\dfrac{\rho^2}{\sigma(\sigma-\rho)}, & \text{exponential}\\[2mm] \dfrac{N-1}{2}+\dfrac{\rho}{\sigma}+\dfrac{\rho^2(2-\sigma)}{2\sigma(\sigma-\rho)}, & \text{deterministic}\\[2mm] \dfrac{N-1}{2}+\dfrac{\rho}{\sigma}+\dfrac{\rho^2(3-\sigma)}{3\sigma(\sigma-\rho)}, & \text{3-stage Erlang}\end{cases} \quad \text{...(28)}$$

$$E(R)=\begin{cases}\dfrac{N-1}{2\lambda}+\dfrac{2b_1}{\sigma}+\dfrac{\rho}{\mu(\sigma-\rho)}+\dfrac{\rho(1-\sigma)}{\sigma\mu(\sigma-\rho)}, & \text{exponential}\\[2mm] \dfrac{N-1}{2\lambda}+\dfrac{2b_1}{\sigma}+\dfrac{\rho}{2\mu(\sigma-\rho)}+\dfrac{\rho(1-\sigma)}{\sigma\mu(\sigma-\rho)}, & \text{determinisitc}\\[2mm] \dfrac{N-1}{2}+\dfrac{2b_1}{\sigma}+\dfrac{2\rho}{3\mu(\sigma-\rho)}+\dfrac{\rho(1-\sigma)}{\sigma\mu(\sigma-\rho)}, & \text{3-stage Erlang}\end{cases}. \quad \text{...(29)}$$

Case 2 : The *M/G/*1 Model with state Dependent Arrival Rate. If $H^*(\theta)\cong B^*(\theta)$ and $N\to 1$, then our model converts to *M/G/*1 model with state dependent arrival rate which was studied by Gupta and Srinivasa Rao (1996a). Their results are derived directly by putting above variable.

Conclusion. In this study, we have considered an optimal N-policy for *M/G/*1 finite queue with Bernoulli feedback. The analytical expressions for the average number of units in the queueing system are established by using generating function and supplementary variable technique in steady state probabilities. The cost analysis is helpful for the system designer in determining the optimal value of threshold parameter so as to minmize total expected cost. The analysis of queueing system operating under N-policy and Bernoulli feedback mechanism, provides unified treatment in many real life congestion problems encountered in computer, communication systems, manufacturing, production and distribution process.

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g-REGULARLY ORDERED SPACES

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ABSTRACT

In the present paper, we introduce the order analogue of g-regular space and prove some results about g-regular space to g-regularly ordered spaces.

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Keywords : Decresaing g-closed, increasing g-open, $T_{1/2}$-ordered, g-regularly ordered.

1. Introduction. A topological ordered space is a topological space equipped with a partial order. The study of topological ordered space was initiated by Nachbin [3]. Nachbin defines a topological ordered spaces to be a topological space if it is also equipped with an order relation. The order may be a preorder (i.e. a reflexive and transitive relation) or a partial order (i.e. an antisymmetric preorder). Every topological spaces may be regarded as a topological ordered space equipped with a discrete (trivial order '$\leqq$', where $x \leqq y$ iff $x=y$. The concept in topological space was introduced by Munshi [2]. In this paper we introduce the order analogue of g-regular space and we prove some results about g-regular space to g-regularly ordered spaces.

2. Preliminary . Let X be a set equipped with a partial order '$\leqq$'. For $y \in X$, the set $\{x \in X : x \leqq y\}$ will be denoted by $[\leftarrow, y]$ and the set $\{x \in X : y \leqq x\}$ will be denoted by $[y, \rightarrow]$. For any subset A of X,

$$i(A) \quad = \quad \cup \quad \{[a, \rightarrow] : a \in A\}, \text{ and}$$

$$\mathrm{d}(A) \quad = \quad \cup \quad \{[\leftarrow, a] : a \in A\}.$$

Obviously, $A \subseteq i(A)$ and $A \subseteq d(A)$, If $A=i(A)$, then A is said to be ***increasing*** and if $A=d(A)$, then A is said to be ***deccreasing***. In other words is A increasing iff $x \leqq y$ and $x \in A \Rightarrow y \in A$, and A is decreasing iff $x \leqq y$ and $y \in A \Rightarrow x \in A$. The complement of a decreasing (increasing) set is increasing (decreasing).

Let $(X, \leqq, T)$ be a topological ordered space. For any subset A of X, let

$D(A) = \cap \{F : F$ is a decreasing closed set containing $A\}$,

$I(A) = \cap \{H : H$ is a increasing closed set containing $A\}$,

$D^0(A) = \cup \{G : G$ is a decreasing open set contained in $A\}$,

$I^0(A) = \cup \{M : M$ is an increasing open set contained in $A\}$.

It can be easily seen that $D(A)$ $(I(A))$ is the smallest decreasing (increasing) closed set containing A and $D^0(A)$ $(I^0(A))$ is the largest decreasing (increasing) open set contained in A.

2.1 Lemma [4]. If A be any subset of a topological ordered space X and if $D(A)$, $I(A)$, $D^0(A)$, $I^0(A)$ be as above, then the following hold :

(i) $X-D(A) = I^0(X-A)$,

(ii) $X-I(A) = D^0(X-A)$,

(iii) $X-I^0(A) = D(X-A)$,

(iv) $X-D^0(A) = I(X-A)$.

3. g-Regularly Ordered Spaces.

3.1 Definition. A subset A of a topological ordered space $(X, \leqq, T)$ is said to be ***decreasing generalized closed*** (written as ***decreasing g-closed***) if $D(A) \subseteq U$ and U is decreasing open in X. Similarly *increasing g-closed* defined dually.

3.2 Definition. A subset A of a topological ordered space $(X, \leqq, T)$ is said to be ***increasing generalized open*** (written as *increasing g-open*) if $X-A$ is decreasing g-closed. Similarly ***decreasing g-open*** defined dually.

Clearly increasing open (resp. decreasing closed) sets are inceasing g-open (resp. decreasing g- closed) sets, but the converse is not necessarily true.

3.3 Definition. A topological ordered space $(X, \leqq, T)$ is called a *lower* (*upper*) $T_{1/2}$*-ordered* if every decreasing (increasing) g-closed set is decreasing (increasing) closed. $(X, \leqq, T)$ is said to be $\boldsymbol{T_{1/2}}$**-ordered** iff $(X, \leqq, T)$ is lower and upper $T_{1/2}$-ordered.

3.4 Theorem. A subset A of a topological ordered space $(X, \leqq, T)$ is increasing (decreasing) g-open iff $F \subseteq I^0(A)(D^0(A))$ whenever $F \subseteq A$ and F is increasing (decreasing) closed.

Proof. Necessary : Let A be an increasing g-open and suppose that $F \subseteq A$ whenever F is an increasing closed set. By definition, $X-A$ is decreasing g-closed set. Also $X-A \subseteq X-F$. Since $X-F$ is decreasing open, therefore $D(X-A) \subseteq X-F$. Hence $X-I^0(A) \subseteq X-F$ and so $F \subseteq I^0(A)$.

Sufficiency : If F is an increasing closed set with $F \subseteq I^0(A)$ whenever $F \subseteq A$, it follows that $X-A \subseteq X-F$ and $X-I^0(A) \subseteq X-F$. Hence $D(X-A) \subseteq X-F$. Thus $X-A$ is decreasing g-closed and so A is increasing g-open. Similarly, the decreasing g-open case may be discussed.

3.5 Definition [1]. A topological ordered space $(X, \leqq, T)$ is said to be ***lower (upper) regularly ordered*** if for each decreasing (increasing) T- closed set $F \subseteq X$ and each element $a \notin F$, there exist disjoint T-neighbourhoods U of a and V of F such that U is increasing (decreasing) and V is decreasing (increasing) in X. $(X, \leqq, T)$ is said to be-**regularly ordered** iff $(X, \leqq, T)$ is both lower and upper regularly ordered.

3.6 Definition. A topological ordered space $(X, \leqq, T)$ is said to be **lower (upper) g–regularly ordered** iff for each decreasing (increasing) g–closed set $F \subseteq X$ and each element $a \notin F$, there exist disjoint T–neighbourhoods U of a and V of F such that U is increasing (decreasing) and V is decreasing (increasing) in X. $(X, \leqq, T)$ is said to be **g-regularly ordered** iff $(X, \leqq, T)$ is both lower and upper regularly ordered.

It follows from the above definition that every g-regularly ordered space is regularly ordered space, but the converse need not be true. Also a space is regularly ordered and $T_{1/2}$- ordered iff it is g-regularly orderd.

3.7 Example. Let $X=\{a,b,c\}$ equipped with the topology $T=\{\phi,\{a\},\{b,c\},X\}$ and with the partial order '$\leqq$' defined as : $a \leqq a$, $b \leqq b$, $b \leqq c$, $c \leqq c$, then $(X, \leqq, T)$ is a g-regularly ordered space.

3.8 Theorem. For a topological ordered space $(X, \leqq, T)$ the following are equivalent:

(a) X is lower (upper) g- regularly ordered.

(b) For each $x \in X$ and every increasing (decreasing) g-open set U containing x, there exists increasing (decreasing) open set V such that

$$x \in V \subseteq I(V)(D(V)) \subseteq U.$$

Proof. $(a) \Rightarrow (b)$. Let U be an increasing (decreasing) g- open set containing x. Then $(X-U)$ is decreasing (incresing) g-closed set such that $x \notin X-U$. It follows that there exists a decreasing (increasing) open set W and an increasing (decreasing) open set V such that $X-U \subseteq W$, $x \in V$ and $W \cap V = \phi$. Since W is decreasing (increasing) open and $V \cap W = \phi$, therefore $W \cap I(V)(D(V)) = \phi$. Thus $x \in V \subseteq I(V)$

$(D(V)) \subseteq X-W \subseteq U$.

$(b) \Rightarrow (a)$. Let F be a decreasing (increasing) g-closed set and $x \notin F$. Then $X-F$ is increasing (decreasing) g-open set such that $x \notin X-F$. By (b), there exists an increasing (decreasing) open set U such that $x \in U \subseteq I(U)(D(U)) \subseteq X-F$. Thus $F \subseteq X-I(U)(D(U))$ which is decreasing (increasing) open and $U \cap (X-I(U)(D(U)) = \phi$. Hence X is g-regularly ordered space.

3.9 Theorem. For a topological ordered space $(X, \leq, T)$, the following are equivalent:

(a) X is lower (upper) g–regularly ordered.

(b) For every decreasing (increasing) g-closed set F, the intersection of all decreasing (increasing) closed decreasing (increasing) neighbourhoods of F is exactly F.

Proof. $(a) \Rightarrow (b)$. Let F be a decreasing (increasing) g–closed subset of X and $x \notin F$. Then $X-F \subseteq U$ is an increasing (decreasing) g-open set containing x, therefore there exists an increasing (decreasing) open set G such that $x \in G \subseteq I(G)(D(G)) \subseteq U$. Hence $F \subseteq X-1(G)(D(G)) \subseteq X-G$ and $x \notin X-G$. Thus X-G is a decreasing (increasing) closed decreasing (increasing) nieghbourhoods of F which does not contains x. Thus the intersection of all decreasing (increasing) closed decreasing (increasing) neighbourhoods of F exactly F.

$(b) \Rightarrow (a)$. Let F be decreasing (increasing) g-closed and $x \notin F$. There exists decreasing (increasing) closed decreasing (increasing) neighbourhood A of F such that $x \notin A$. A is decreasing (increasing) closed decreasing (increasing) neighbourhood of F implies that there exists a decreasing (increasing) open set V such that $F \subseteq V \subseteq A$. Now, $x \in X-A$ which is increasing (decreasing) open and $F \subseteq V$ which is decreasing (increasing) open such that $V \cap X-A = \phi$.

3.10 Theorem. A topological ordered space $(X, \leq, T)$ is g-regularly ordered iff for every set A and every increasing (decreasing) g-open set B such that $A \cap B \neq \phi$, there exists an increasing (decreasing) open set G such that $A \cap G \neq \phi$, and $I(G)(D(G)) \subseteq B$.

Proof. Let $A \subseteq X$ and B be an increasing (descreasing) g–open such that $A \cap B \neq \phi$. Let $x \in A \cap B$. Then there exists an increasing (decreasing) open set G such that $x \in G \subseteq I(G)(D(G)) \subseteq B$. Clearly, $G \cap A \neq \phi$ as $x \in G \cap A$ and $(G)(D(g)) \subseteq B$.

Conversely, let F be a decreasing (increasing) g-closed set and $x \notin F$. If $A = \{x\}$ and $B = X-F$, then $A \cap B \neq \phi$. There exists an increaisng (decreasing) open set G such

that $A \cap G \neq \phi$ and $I(G)(D(G)) \subseteq B$. $F=X-B \subseteq X-I(G)(D(G))$. Thus $x \in G$, $F \subseteq X-I(G)(D(G))$, G is an increasing (decreasing) open, $X-I(G)(D(G))$ is decreasing (increasing) open and $G \cap (X-I(G)(D(G)) = \phi$.

3.11 Theorem. A topological ordered space $(X, \leqq, T)$ is g-regularly ordered iff for every non-empty set A and any decreasing (increasing) g-closed set B satisfying $A \cap B = \phi$, there exists disjoint open sets G and H such that $A \cap G \neq \phi$ and $B \subseteq H$, where G is increasing (decreasing) open and H is decreasing (increasing) open.

Proof. $A \cap B = \phi \Rightarrow A \cap (X - B) \neq \phi$. Therefore, there exists an increasing (decreasing) open set G such that $A \cap G \neq \phi$ and $I(G)(D(G)) \subseteq X-B$. Then G and $H=X-1(G)(D(G))$ satisfying the required property.

Conversely, let F be decreasing(increasing) g–closed set and $x \notin F$. Put $A=\{x\}$ and $B=F$. Then there exists an increasing (decreasing) open set G and decreasing (increasing) open set H such that $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$. Hence X is g regularly ordered.

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that $A \cap G \neq \phi$ and $I(G)(D(G)) \subseteq B$. $F = X - B \subseteq X - I(G)(D(G))$. Thus $F \subseteq G^1$, $F \subseteq X - I(G)(D(G))$, G is an increasing (decreasing) open and $X - I(G)(D(G))$ is decreasing (increasing) open and $G^1 \cap (X - I(G)(D(G))) = \phi$.

3.11 Theorem. A topological ordered space $(X, \leq, T)$ is g-regularly ordered iff for every non-empty set A and any decreasing (increasing) g-closed set B satisfying $A \cap B = \phi$, there exist disjoint open sets G and H such that $A \cap G \neq \phi$ and $B \subseteq H$, where G is increasing (decreasing) open and H is decreasing (increasing) open.

Proof. $A \cap B = \phi \Rightarrow A \cap (X - B) \neq \phi$. Therefore, there exists an increasing (decreasing) open set G such that $A \cap G \neq \phi$ and $I(G)(D(G)) \subseteq X - B$. Then G and $H = X - I(G)(D(G))$ satisfying the required property.

Conversely, let F be decreasing (increasing) g-closed set and $x \notin F$. Let $A = \{x\}$ and $B = F$. Then there exists an increasing (decreasing) open set G and decreasing (increasing) open set H such that $A \subseteq G$, $B \subseteq H$ and $G \cap H = \phi$. Hence X is g-regularly ordered.

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ON THE DEGREE OF APPROXIMATION OF EULER'S MEANS

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ABSTRACT

In the present paper we obtain the degree of approximation using the Euler's means of function belonging to the generalized Hölder metric and two corollaries are also obtained.

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Keywords : Generalized Hölder metric, Banach space.

1. Introduction. Let $C_{2\pi}$ be the space of all periodic functions f on $[0, 2\pi]$ with the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \sum_{n=0}^{\infty} A_k(x) \qquad ...(1.1)$$

we defined the space H_w by

$$H_w = \{f \varepsilon C_{2\pi} : |f(x) - f(f)| \le k\omega(|x-y|)\} \qquad ...(1.2)$$

and the norm $\|\cdot\|_w$. by

$$\|f\|_{w^\cdot} = \|f\|_C + \sup_{x,y}\{\Delta^{w^*} f(x,y)\} \qquad ...(1.3)$$

where $\|f\|_C = \sup_{0 \le x \le 2\pi} |f(x)|$,

and $$\Delta^{w^*} f(x,y) = \frac{|f(x) - f(y)|}{w^*(|x-y|)}, x \ne y \qquad ...(1.4)$$

$\Delta^0 f(x,y) = 0$, $w(t)$, $w^*(t)$ being increasing functions of t.

If $w(|x-y|) \le A|x-y|^\alpha$ and $w^*(|x-y|) \le k|x-y|^\beta$, $\qquad$ ...(1.5)

$0 < \alpha \le 1, 0 \le \beta < \alpha$ A and k being positive constants, then the space

$$H_\alpha = \{f \varepsilon C_{2\pi} : |f(x) - f(y)| \le k|x-y|^\alpha\}, 0 < \alpha \le 1$$

is a Banach space [1] and the metric induced by the norm $\|.\|_\alpha$ on H_α is said to be Hölder metric.

Let $E_n^q(f;x)$ be the Euler's (E,q)-means for $q>0$ defined by

$$E_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} S_k(f;x) \qquad \text{...(1.6)}$$

where

$$S_k(f;x) = \frac{1}{2\pi}\int_0^{\pi} f(x+t)\frac{\sin(k+1/2)t}{\sin t/2}$$

is the k^{th} partial sum of Fourier Sereis (1.1).

we write

$$\phi_x(t) = f(x+t) + f(x-t) - 2f(x). \qquad \text{...(1.7)}$$

2. Previous Results. Mahapatra and Chandra [2] in the year 1983 obtained the degree of approximation of Euler transform of the Fourier series of t in the Hölder metric.

Theorem *A*-Let $0 \le \beta < \alpha \le 1$. Then for $f \varepsilon H_\alpha$,

$$\left\|E_n^q(f) - f\right\|_\beta = O\left\{n^{(\alpha-\beta)/2}(\log n)^{\beta/\alpha}\right\}, \qquad \text{...(2.1)}$$

where $E_n^q(f;x)$ is the Euler mean of the Fourier series.

Then Chandra [3] gave a better result of above theorem A in 1988.

Thorem *B*-Let $0 \le \beta < \alpha \le 1$ and let $f \varepsilon H_\alpha$. Then

$$\left\|E_n^q(f) - f\right\|_\beta = O\left\{n^{(\beta-\alpha)}\log n\right\}. \qquad \text{...(2.2)}$$

The object of this paper is to obtain the degree of approximation on the generalised Hölder metric, using the $(E,q)(q>0)$-mean.

3. Main Result. Theorem -Let $f \in H_w$, $0 < n \le 1$, $0 \le \beta < \eta$ and let $w(t)$ be the modulus of continuity, then

$$\left\|E_n^q(f,x) - f\right\|_{w^*} = O\left[\log n (w(\pi/n))^{1-\beta/\eta}\right]. \qquad \text{...(3.1)}$$

For the proof of the theorem, we require the following lemmas :

4. Lemma.

(i) $\quad Q_n(t) = O(n) \qquad 0 < t < \pi/n \qquad$ ...(4.1)

(ii) $\quad Q_n(t) = O(t^{-1}) \qquad \pi/n < t < \pi \qquad$ ...(4.2)

Proof : It is given that

$$Q_n(t) = (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \frac{\sin(k+1/2)t}{2\sin t/2} \qquad \text{...(4.3)}$$

Using the estimates

$|\sin(k+1/2)t|t \le (k+1/2)t$ and $|\sin(k+1/2)|t \le 1$

respectively, for the proof of lemma 4(i) and Lemma 4(ii) with the fact that

$$(1+q)^{n-1} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} = 1,$$

we get

$$|Q_n(t)| = O(1) \begin{cases} (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} (k+1/2)t/t \\ (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} (1/t) \end{cases} = O(1) \begin{cases} (n)(1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \\ (t^{-1})(1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \end{cases}.$$

Hence,

$$Q_n(t) = O(n) \quad \text{and} \quad Q_n(t) = O(t^{-1}).$$

5. Proof of the Theorem. It is known that by (1.6)

$$E_n^q(f;x) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} S_k(f;x)$$

and let

$$\sigma_n(x) = E_n^q(f;x) - f(x) = \frac{1}{2\pi(1+q)^n} \int_0^\pi \frac{\phi_x(t)}{\sin t/2} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t dt$$

Similarly

$$\sigma_n(y) = E_n^q(f;y) - f(y) = \frac{1}{2\pi(1+q)^n} \int_0^\pi \frac{\phi_y(t)}{\sin t/2} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t dt$$

and

$$\sigma_n(x;y) = |\sigma_n(x) - \sigma_n(y)| = \frac{1}{2\pi(1+q)^n} \int_0^\pi \frac{\phi_x(t) - \phi_y(t)}{\sin t/2} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t dt$$

$$= \frac{1}{2\pi(1+q)^n}\left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right] \frac{|\phi_x(t)-\phi_y(t)|}{\sin t/2} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin\left(k+\frac{1}{2}\right)t\, dt$$

$$= I_1 + I_2 (\text{Say}). \tag{5.1}$$

Now,

$$|\phi_x(t)-\phi_y(t)| = |\{f(x+t)+f(x-t)-2f(x)\} - \{f(y+t)-f(y-t)-2f(y)\}|$$

$$\le |f(x+t)-f(x)| + |f(x)-f(x-t)| + |f(y+t)-f(y)| + |f(y)-f(y-t)|$$

$$\le kw(t)+kw(t)+kw(t)+kw(t)$$

$$\le 4kw(t) \tag{5.2}$$

$$|\phi_x(t)-\phi_y(t)| \le |f(x+t)-f(y+t)| + |f(x-t)-f(y-t)| + 2|f(x)-f(y)|$$

$$\le kw(|x+t-y-t|) + kw(|x-t-y+t|) + 2kw(|x-y|)$$

$$\le 4kw(|x-y|). \tag{5.3}$$

From (5.2)

$$I_1 = \frac{1}{2\pi(1+q)^n}\int_0^{\pi/n} \frac{|\phi_x(t)-\phi_y(t)|}{\sin t/2} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t\, dt$$

$$= O(1)\int_0^{\pi/n} w(t) Q_n(t) dt \qquad \text{by (4.3) and (5.2)}$$

$$= O(n)\int_0^{\pi/n} w(t)\, dt \qquad \text{by (4.1)}$$

$$= O[nw(\pi/n)(\pi/n)]$$

$$= O(w(\pi/n)). \tag{5.4}$$

Consider,

$$I_2 = \frac{1}{2\pi(1+q)^n}.\int_{\pi/n}^{\pi} \frac{|\phi_x(t)-\phi_y(t)|}{\sin t/2} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t\, dt$$

$$= O(1)\int_{\pi/n}^{\pi} w(t)(1/t) dt \qquad \text{by (4.2)}$$

$$= O[w(\pi/n)][\log t]_{\pi/n}^{\pi}$$

$$= O[\log n \;\; w(\pi/n)]. \tag{5.5}$$

Combining (5.4) and (5.5)

$$I = O[\log n\, w(\pi/n)]$$

Again using (5.3),

$$I_1 = \int_0^{\pi/n} \frac{|\phi_x(t) - \phi_y(t)|}{\sin t/2} (1+q)^x \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t\, dt$$

$$= O[n\, w(|x-y|)(\pi/n)]$$

$$= O[w(|x-y|)] \tag{5.6}$$

and

$$I_2 = \int_{\pi/n}^{\pi} \frac{|\phi_x(t) - \phi_y(t)|}{\sin t/2} (1+q)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \sin(k+1/2)t\, dt$$

$$= O[\log_n w(|x-y|)]\,. \tag{5.7}$$

Combining (5.6) and (5.7) we get,

$$I = O[\log n\, w(|x-y|)]\,.$$

Writing,

$$I = I^{1-\beta/\eta} I^{\beta/\eta}$$

$$I = O[\log n\, w(\pi/n)]^{1-\beta/\eta} [\log n\, w|x-y|]^{\beta/\eta}$$

$$= O\left[\log n (w(\pi/n))^{1-\beta/\eta} (w(|x-y|))^{\beta/\eta}\right].$$

Thus,

$$\sup_{x,y} |\Delta^{w^*} \sigma_n(x;y)| = \sup_{x,y} \frac{|\sigma_n(x) - \sigma_n(y)|}{w^*(|x-y|)}\,]$$

$$= \frac{O\left[\log n\, w(\pi/n)^{1-\beta/\eta}\right] O(w|x-y|)^{\beta/\eta}}{w^*(|x-y|)}$$

$$= O\left[\log n\, (w(\pi/n))^{1-\beta/\eta}\right].$$

Finally

$$\|\sigma_n(x)\|_C = \max_{0\le x\le 2\pi} |E_n^q(f;x) - f(x)| = O(w(\pi/n)).$$

Thus, collecting the above estimates, we have

$$\left\|E_n^q(f;x)-f(x)\right\|_{w^*}=O\left[\log n(w(\pi/n))^{1-\beta/\eta}\right]$$

This completes the proof.

Corollaries . If we put $\eta=\alpha, w(|x-y|)\le A|x-y|^\alpha, w^*(|x-y|)\le k|x-y|^\beta$.

Corollary 1. For $f\varepsilon H_\alpha$, $0<\alpha\le 1$, $0\le\beta<\alpha$ for all $x\varepsilon[0,2\pi]$ then

$$\left\|E_n^q(f;x)-f(x)\right\|=\begin{bmatrix} O\left(n^{\beta-\alpha}(\log n)^{\beta/\alpha}\right) & 0<\alpha<1 \\ O\left(n^{\beta-1}(\log n)^{\beta}\right) & \alpha=1 \end{bmatrix}$$

Corollary 2. Let $f\varepsilon$ Lip α; $0<\alpha\le 1$, then

$$\left\|E_n^q(f;x)-f(x)\right\|=O(1)\left(n^{-\alpha}\log n\right)$$

Proof put $\beta=0$ in the above corallary.

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MULTIVARIABLE ANALOGUES OF GENERALIZED TRUESDELL POLYNOMIALS

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ABSTRACT

In the present paper, we introduce a generalization of multivariable analogues of generalized Truesdell polynomials, due to Chauhan ([5], p.112, (6.1.3). Our polynomials may also be regarded as multivariable analogues of generalized Truesdell polynomials due to Chandel ([1],[2],[3],[4]). In the last, we also introduce further generlization of our polynomials.

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1. Introduction. Singh [6] introduced Truesdell polynomials defined by Rodrigues' formula :

$$T_n^{\alpha}(x,r,p)=x^{-\alpha}e^{px^r}\delta^n\left\{x^{\alpha}e^{-px^r}\right\},\ \delta\equiv x\frac{d}{dx}. \tag{1.1}$$

Chandel ([1],[2],[3],[4]) studied a class of polynomials defined by Rodrigues' formula :

$$T_n^{(\alpha,k)}(x,r,p)=x^{-\alpha}e^{px^r}\Omega_x^n\left\{x^{\alpha}e^{-px^r}\right\},\ \Omega_x\equiv x^k\frac{d}{dx}, \tag{1.2}$$

where $k\neq 1$.

For $k\to 1$, (1.2) reduces to (1.1).

Recently, Chauhan ([5],p.112, (6.1.3) introduced and studied a multivariable analogue of generlaized Truesdell polynomials defined by Rodrigues' formula

(1.3) $T_{n_1,\dots,n_m}^{(b;\alpha_1,\dots,\alpha_n;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$

$$=\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\dots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^b$$

$$\delta_1^{n_1}\dots\delta_m^{n_m}\left\{\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\dots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-b}\right\},$$

where n_i are positive integers, a_i , r_i, p_i, b are arbitrary numbers real or complex independent of all variables x_i; $\delta_i = x_i \dfrac{\partial}{\partial x_i}$, $i = 1,\ldots,m$.

It is clear that

$$(1.4)\quad \lim_{b\to\infty} T_{n_1,\ldots,n_m}^{\left(b;\frac{\alpha_1}{b},\ldots,\frac{\alpha_m}{b};r_1,\ldots,r_m;\frac{p_1}{b},\ldots,\frac{p_m}{b}\right)}(x_1,\ldots,x_m)$$

$$= T_{n_1}^{\alpha_1}(x,r,p_1)\ldots T_{n_m}^{\alpha_m}(x_m,r_m;p_m)\ .$$

Motivated by (1.2) and (1.3), here in the present paper we introduce multivariable analogue of Chandel polynomials ([1],[2],[3],[4])

$$\left\{T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;k_1,\ldots,k_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)/\,n_i = 1,2,\ldots;i = 1,\ldots,m\right\}$$

defined through Rodrigues' formula:

$$(1.5)\quad T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;k_1,\ldots,k_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= \left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right)-\ldots-\left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{b}$$

$$\Omega_{x_1}^{n_1}\ldots\Omega_{x_m}^{n_m}\left\{\left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right)-\ldots-\left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{-b}\right\};$$

where n_i are positive integers, b, α_i, $k_i \neq 1$, r_i, p_i are arbitrary numbers real or complex independent of all variables $x_i;\Omega x_i \equiv x_i^{k_i}\dfrac{\partial}{\partial x_i}, i = 1,\ldots,m.$

For $k_i \to 1$ $(i=1,\ldots,m)$, (1.5) reduces to (1.3).

From (1.5) and (1.2), it is also clear that

$$(1.6)\quad \lim_{b\to\infty} T_{n_1,\ldots,n_m}^{\left(b;\frac{\alpha_1}{b},\ldots,\frac{\alpha_m}{b};k_1,\ldots,k_m;r_1,\ldots,r_m;\frac{p_1}{b},\ldots,\frac{p_m}{b}\right)}(x_1,\ldots,x_m)$$

$$= T_{n_1}^{(\alpha_1,k_1)}(x_1,r_1,p_1)\ldots T_{n_m}^{(\alpha_m,k_m)}(x_m,r_m,p_m)$$

where $T_n^{(\alpha,k)}(x,r,p)$ are generalized Truesdell polynomials due to Chandel ([1],[2],[3],[4]) defined through Rodrigues' formula (1.2).

For brevity, we shall write

$$T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;k_1,\ldots,k_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

as

$T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$.

2. Generating Relation. Starting with Rodrigues' formula (1.5), we have

$$\sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;[\alpha],[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$= \left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right)-\ldots-\left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^b \exp\left(t_1\Omega_{x_1}+\ldots+t_m\Omega_{x_m}\right)$$

$$\left\{\left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right)-\ldots-\left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{-b}\right\}.$$

Now applying the familiar result due to Chandel ([1, p.105 eq. (5.2.5); Also see Srivastava-Singhal [7,p.76 eq. (1.12)]

$$(2.1)\quad e^{t\Omega_x}\{f(x)\} = f\left(\frac{x}{\left\{1-(k-1)tx^{k-1}\right\}^{\frac{1}{(k-1)}}}\right),$$

where $k \neq 1$ and $f(x)$ admits Taylor's series expansion, we have

$$\sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_n^{n_m}}{n_m!}$$

$$= \left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right)-\ldots-\left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^b$$

$$\left[1-\left(\alpha_1 \log\frac{x_1}{\left\{1-(k_1-1)t_1 x_1^{k_1-1}\right\}^{\frac{1}{(k_1-1)}}} - \frac{p_1 x_1^{r_1}}{\left(1-(k_1-1)t_1 x_1^{k_1-1}\right)^{\frac{r_1}{(k_1-1)}}}\right)\right.$$

$$\left.-\ldots-\left(\alpha_m \log\frac{x_m}{\left[1-(k_m-1)t_m x_m^{k_m-1}\right]^{\frac{r_m}{(k_m-1)}}} - \frac{p_m x_m^{r_m}}{\left[1-(k_m-1)t_m x_m^{k_m-1}\right]^{\frac{r_m}{(k_m-1)}}}\right)\right]^{-b}$$

Therefore, we finally derive the generating relation

$$(2.2)\quad \sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$= \left[1 - \left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \ldots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^b$$

$$\left[1 - (\alpha_1 \log x_1 + \ldots + \alpha_m \log x_m) + \frac{\alpha_1}{(k_1 - 1)} \log\left\{1 - (k_1 - 1) t_1 x_1^{k_1 - 1}\right\}\right.$$

$$+ \ldots + \frac{\alpha_m}{(k_m - 1)} \log\left\{1 - (k_m - 1) t_m x_m^{k_m - 1}\right\} + p_1 x_1^{r_1} \left\{1 - (k_1 - 1) t_1 x_1^{k_1 - 1}\right\}^{-\frac{r_1}{k_1 - 1}}$$

$$\left. + \ldots + p_m x_m^{r_m} \left\{1 - (k_m - 1) t_m x_m^{t_m - 1}\right\}^{-\frac{r_m}{k_m - 1}}\right]^{-b}.$$

3. Application of Generating Relation. Making an appeal to generating relation (2.2), we have

$$\sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!}$$

$$= \sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!} \sum_{s_1,\ldots,s_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b';[\alpha];[k];[r];[p])}(x_1,\ldots,x_m) \frac{t_1^{s_1}}{s_1!} \cdots \frac{t_m^{s_m}}{s_m!}$$

Therefore, we finally derive

(3.1) $T_{n_1,\ldots,n_m}^{(b+b';[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$

$$= \sum_{s_1=0}^{n_1} \cdots \sum_{s_m=0}^{n_m} \binom{n_1}{s_1} \cdots \binom{n_m}{s_m} T_{n_1-s_1,\ldots,n_m-s_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m) T_{s_1,\ldots,s_m}^{(b';[\alpha];[k];[r];[p])}(x_1,\ldots,x_m),$$

which can be further generalized in the form

(3.2) $T_{n_1,\ldots,n_m}^{(b_1+\ldots+b_q;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$

$$= \sum_{s_{11}+\ldots+s_{1q}=n_1} \cdots \sum_{s_{m1}+\ldots+s_{mq}=n_m} \prod_{j=1}^{q} \binom{n_1}{s_{ij}} \cdots \binom{n_m}{s_{mj}} T_{s_{11},\ldots,s_{m1}}^{(b_1;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m) \cdots$$

$$T_{s_{1q},\ldots,s_{mq}}^{(b_q;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m).$$

4. Recurrence Relations. By making an appeal to (2.2) we have

$$\left[1 - \alpha_1 \log x_1 - \ldots - \alpha_m \log x_m + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m}\right] \sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m) \frac{t_1^{s_1}}{s_1!} \cdots \frac{t_m^{s_m}}{s_m!}$$

$$= \sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!} \left[1 - (\alpha_1 \log x_1 + \ldots + \alpha_m \log x_m)\right.$$

$$-\frac{\alpha_1}{k_1}\sum_{s_1=0}^{\infty}(k_1-1)^{s_1}\frac{x_1^{(k_1-1)s_1}}{s_1!}t_1^{s_1}\ldots-\frac{\alpha_m}{k_m}\sum_{s_m=0}^{\infty}(k_m-1)^{s_m}x_m^{(k_m-1)s_m}\frac{t_m^{s_m}}{s_m!}+p_1x_1^{r_1}\sum_{s_1=0}^{\infty}\frac{(r_1/(k_1-1))_{s_1}}{s_1!}$$

$$(k_1-1)^{s_1}x_1^{(k_1-1)s_1}t_1^{s_1}+\ldots+p_mx_m^{r_m}\sum_{s_m=0}^{\infty}\frac{(r_m/(k_m-1))_{s_m}}{s_m!}(k_m-1)^{s_m}x_m^{(k_m-1)s_m}t_m^{s_m}\Bigg].$$

Now equating the coefficiants of $t_1^{n_1}\ldots t_m^{n_m}$ both the sides, we derive the recurrence relation

(4.1) $$\left(1-\alpha_1\log x_1-\ldots-\alpha_m\log x_m+p_1x_1^{r_1}+\ldots+p_mx_m^{r_m}\right)$$

$$=\left[1-(\alpha_1\log x_1+\ldots+\alpha_m\log x_m)\right]T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$+\sum_{s_1=0}^{n_1}\binom{n_1}{s_1}\left(p_1x_1^{r_1}(r_1/(k_1-1))_{s_1}-\alpha_1/(k_1-1)\right)(k_1-1)^{s_1}x_1^{(k_1-1)s_1}T_{n_1-s_1,n_2,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$+\ldots+\sum_{s_m=0}^{n_m}\binom{n_m}{s_m}\left[p_mx_m^{r_m}(r_m/(k_m-1))_{s_m}-\alpha_m/(k_m-1)\right](k_m-1)^{s_m}x_m^{(k_m-1)s_m}$$

$$T_{n_1,\ldots,n_{m-1},n_m-s_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m).$$

Differentiating (2.2) partially with respect to t_1, we have

$$\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\ldots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]$$

$$\sum_{n_1,\ldots,n_m=0}^{\infty}T_{n_1,\ldots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1-1}}{(n_1-1)!}\frac{t_2^{n_2}}{n_2!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=-b\sum_{n_1,\ldots,n_m=0}^{\infty}T_{n_1,\ldots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$\left[-\alpha_1x_1^{k_1-1}\sum_{s_1=0}^{\infty}\left\{(k_1-1)t_1x_1^{k_1-1}\right\}^{s_1}+p_1r_1x_1^{r_1+k_1-1}\sum_{s_1=0}^{\infty}t_1^{s_1}\left(\frac{r_1}{k_1-1}+1\right)_{s_1}\left\{(k_1-1)x_1^{k_1-1}\right\}^{s}\right].$$

Thus equating the coefficients of $t_1^{n_1}\ldots t_m^{n_m}$ both the sides, we derive the recurrence relation

(4.2) $$\left[1-\alpha_1\log x_1-\ldots-\alpha_m\log x_m+p_1x_1^{r_1}+\ldots+p_mx_m^{r_m}\right]T_{n+1,n_2,\ldots,n_n}^{(b;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$=b\alpha_1x_1^{k_1-1}\sum_{s_1=0}^{n_1}\left\{(k_1-1)x_1^{k_1-1}\right\}^{s_1}s_1!\binom{n_1}{s_1}T_{n_1-s_1,n_2,\ldots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$- bp_1 r_1 x_1^{r_1+k_1-1} \sum_{s_1=0}^{n_1} \binom{n_1}{s_1} \left(\frac{r_1}{k_1 - 1} + 1 \right)_{s_1} \left[(k_1 - 1) x_1^{k_1 - 1} \right]^{s_1} T^{(b+1;[\alpha];[k];[r];[p])}_{n_1 - s_1, n_2, \ldots, n_m}(x_1, \ldots, x_n)$$

which suggests that m-recurrence relations can be expressed in the following unified form :

$$(4.3) \left[1 - \alpha_1 \log x_1 - \ldots - \alpha_m \log x_m + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m} \right] T^{(b;[\alpha];[k];[r];[p])}_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_m}(x_1, \ldots, x_m)$$

$$= b\alpha_i x_i^{k_i - 1} \sum_{s_i=0}^{n_i} s_i! \binom{n_i}{s_i} \left[(k_i - 1) x_i^{k_i - 1} \right]^{s_i} T^{(b+1;[\alpha];[k];[r];[p])}_{n_1,\ldots,n_{i-1},n_i - s_i, n_{i+1},\ldots,n_m}(x_1, \ldots, x_m)$$

$$- bp_i r_i x_i^{r_i + k_i - 1} \sum_{s_i=0}^{n_i} \binom{n_i}{s_i} \left(\frac{r_i}{k_i - 1} + 1 \right)_{s_i} \left[(k_i - 1) x_i^{k_i - 1} \right]^{s_i} T^{(b+1;[\alpha];[k];[r];[p])}_{n_1,\ldots,n_{i-1},n_i - s_i, n_{i+1},\ldots,n_m}(x_1, \ldots, x_m).$$

Differentiating (2.2) partially with respect to x_1, we have

$$\left[1 - \alpha_1 \log x_1 - \ldots - \alpha_m \log x_m + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m} \right]$$

$$\sum_{n_1,\ldots,n_m=0}^{\infty} \frac{\partial}{\partial x_1} T^{(b;[\alpha];[k];[r];[p])}_{n_1,\ldots,n_m}(x_1, \ldots, x_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!}$$

$$= b\left(-\alpha_1 / x_1 + p_1 r_1 x_1^{r_1 - 1} \right) \sum_{n_1,\ldots,n_m=0}^{\infty} T^{(b;[\alpha];[k];[r];[p])}_{n_1,\ldots,n_m}(x_1, \ldots, x_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!}$$

$$+ b \sum_{n_1,\ldots,n_m=0}^{\infty} T^{(b+1;[\alpha];[k];[r];[p])}_{n_1,\ldots,n_m}(x_1, \ldots, x_m) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_m^{n_m}}{n_m!} \left[\frac{\alpha_1}{x_1} + \alpha_1 (k_1 - 1) x_1^{k_1 - 2} t_1 \right.$$

$$\left. \sum_{s_1=0}^{\infty} \left[(k_1 - 1) x_1^{k_1 - 1} \right]^{s_1} t_1^{s_1} - p_1 x_1^{r_1 + k_1 - 2} r_1 k (k_1 - 1) t_1 \sum_{s_1=0}^{\infty} \frac{(r_1 / k_1 + 1)_{s_1}}{s_1!} \left[(k_1 - 1) x_1^{k_1 - 1} \right]^{s_1} t_1^{s_1} \right].$$

Now equating the coefficients of $t_1^{n_1} \ldots t_m^{n_m}$ both the sides, we establish

$$(4.4) \left[1 - \alpha_1 \log x_1 - \ldots - \alpha_m \log x_m + p_1 x_1^{r_1} + \ldots + p_m x_m^{r_m} \right]$$

$$\frac{\partial}{\partial x_1} T^{(b;[\alpha];[k];[r];[p])}_{n_1,\ldots,n_n}(x_1, \ldots, x_m)$$

$$= b\left(-\frac{\alpha_1}{x_1} + p_1 r_1 x_1^{r_1 - 1} \right) T^{(b;[\alpha];[k];[r];[p])}_{n_1,\ldots,n_m}(x_1, \ldots, x_m) + b\left[\frac{\alpha_1}{x_1} T^{(b+1;[\alpha];[k];[r];[p])}_{n_1,\ldots,n_m}(x_1, \ldots, x_m) \right.$$

$$+\alpha_1(k_1-1)x_1^{k_1-2}\sum_{s_1=0}^{n_1-1}\left[(k_1-1)x_1^{k_1-1}\right]^{s_1}\frac{n_1!}{(n_1-1-s_1)!}T_{n_1-s_1-1,n_2,\dots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$-p_1x_1^{r_1+k_1-2}r_1(k_1-1)\sum_{s_1=0}^{n_1-1}\binom{n_1-1}{s_1}\left(\frac{r_1}{k_1}+1\right)_{s_1}\left[(k_1-1)x_1^{k_1-1}\right]^{s_1}T_{n_1-s_1-1,n_2,\dots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\dots,x_m)\Bigg],$$

which further suggests on-recurrence relations in the following unified form :

(4.5) $$\left(1-\alpha_1\log x_1-\dots-\alpha_m\log x_m+p_1x_1^{r_1}+\dots+p_mx_m^{r_m}\right)\frac{\partial}{\partial x_i}T_{n_1,\dots,n_m}^{(b;[\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$=b\left(-\frac{\alpha_i}{x_i}+p_ir_ix_i^{r_i-1}\right)T_{n_1,\dots,n_n}^{(b;[\alpha];[k];[r];[p])}(x_1,\dots,x_m)+b\Bigg[\frac{\alpha_i}{x_i}T_{n_1,\dots,n_n}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$+\alpha_i(k_i-1)x_i^{k_i-2}\sum_{s_i=0}^{n_i-1}\left[(k_i-1)x_i^{k_i-1}\right]^{s_i}\frac{n_i!}{(n_i-1-s_i)!}T_{n_1,\dots,n_{i-1},n_i-s_i-1,n_{i+1},\dots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\dots,x_n)$$

$$-p_ix_i^{r_i+k_i-2}r_i(k_i-1)\sum_{s_i=0}^{n_i-1}\binom{n_i-1}{s_i}\left(\frac{r_i}{k_i}+1\right)_{s_i}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i}$$

$$T_{n_1,\dots,n_{i-1},n_i-s_i-1,n_{i+1},\dots,n_m}^{(b+1;[\alpha];[k];[r];[p])}(x_1,\dots,x_n)\Bigg],\qquad i=1,\dots,m.$$

5. Generalization. Consider

(5.1) $$G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$=\left[G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-1}$$

$$\Omega_{x_1}^{n_1}\dots\Omega_{x_m}^{n_m}\left\{G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right\},$$

where

(5.2) $$G(z)=\sum_{n=0}^{\infty}\gamma_nz^n,\quad \gamma_0\neq 0.$$

For $\gamma_n=(b)_n/n!$, (5.1) reduces to (1.5)

For $\gamma_n=1/n!$, (5.1) defines

(5.3) $$E_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)$$

$$=T_{n_1}^{(\alpha_1,k_1)}(x_1,r,p)\dots T_{n_m}^{(\alpha_m,k_m)}(x_m,r_m,p_m),$$

where $T_n^{(\alpha,k)}(x,r,p)$ are polynomials due to Chandel ([1],[2],[3],[4]) defined by (1.2)

6. Generating Relation. Starting with Rodrigues' formula (5.1), we have

$$\sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-1}$$

$$e^{t_1\Omega_{x_1}+\dots+t_m\Omega_{x_m}}\left\{G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right\}$$

$$=\left[G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-1}$$

$$G\left[\alpha_1\log\left(\frac{x_1}{\left\{1-(k_1-1)t_1x_1^{k_1-1}\right\}^{1/(k_1-1)}}\right)-\frac{p_1x_1^{r_1}}{\left\{1-(k_1-1)t_1x_1^{k_1-1}\right\}^{r_1/(k_1-1)}}\right.$$

$$\left.+\dots+\alpha_m\log\left(\frac{x_m}{\left\{1-(k_m-1)t_mx_m^{k_m-1}\right\}^{1/(k_m-1)}}\right)-\frac{p_mx_m^{r_m}}{\left\{1-(k_m-1)t_mx_m^{k_m-1}\right\}^{r_m/(k_m-1)}}\right].$$

Thus we derive the generating relation

$$(6.1)\quad \sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-1}$$

$$G\left[\alpha_1\log\left(\frac{x_1}{\left\{1-(k_1-1)t_1x_1^{k_1-1}\right\}^{1/(k_1-1)}}\right)-\frac{p_1x_1^{r_1}}{\left\{1-(k_1-1)t_1x_1^{k_1-1}\right\}^{r_1/(k_1-1)}}\right.$$

$$\left.+\dots+\alpha_m\log\left(\frac{x_m}{\left\{1-(k_m-1)t_mx_m^{k_m-1}\right\}^{1/(k_m-1)}}\right)-\frac{p_mx_m^{r_m}}{\left\{1-(k_m-1)t_mx_m^{k_m-1}\right\}^{r_m/(k_m-1)}}\right].$$

7. Recurrence Relations. Differenting (6.1), partially with respect to t_i, we have

$$(7.1)\quad \sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_{i-1}^{n_{i-1}}}{n_{i-1}!}\frac{t_i^{n_i-1}}{(n_i-1)!}\frac{t_{i+1}^{n_{i+1}}}{n_{i+1}!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[G\left(\alpha_1\log x_1-p_1x_1^{r_1}+\dots+\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-1}G$$

$$\left[\frac{\alpha_i x_i^{k_i-1}}{1-(k_i-1)t_i x_i^{k_i-1}} - p_i r_i x_i^{r_i+k_i-1}\left\{1-(k_i-1)t_i\left(\frac{-r_i}{k_i-1}-1\right)\right\}\right]$$

$$= \left[G\left(\alpha_1 \log x_1 - p_1 x_1^{r_1} + \dots + \alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{-1}.G'$$

$$\left[\alpha_i x_i^{k_i-1}\sum_{s_i=0}^{\infty}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i} t_i^{s_i} - r_i k_i x_i^{r_i+k_i-1}\sum_{s_i=0}^{\infty}\frac{\left(\frac{r_i}{k_i-1}+1\right)_{s_i}}{s_i!}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i} t_i^{s_i}\right].$$

Similarly differentiating (6.1) partially with respect to t_j, we have

$$\text{(7.2)}\qquad \sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_{j-1}^{n_{j-1}}}{n_{j-1}!}\frac{t_j^{n_j-1}}{(n_j-1)!}\frac{t_{j+1}^{n_{j+1}}}{n_{j+1}!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$= \left[G\left(\alpha_1 \log x_1 - p_1 x_1^{r_1} + \dots + \alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{-1}.G'$$

$$\left[\alpha_j x_j^{k_j-1}\sum_{s_j=0}^{\infty}\left\{(k_j-1)x_j^{k_j-1}\right\}^{s_j} t_j^{s_j} - r_j p_j x_j^{r_j+k_j-1}\sum_{s_j=0}^{\infty}\frac{\left(\frac{r_j}{k_j-1}+1\right)_{s_j}}{s_j!}\left\{(k_j-1)x_j^{k_j-1}\right\}^{s_j} t_j^{s_j}\right].$$

Now eleminating G' from (7.1) and (7.2), we obtain

$$\left[\alpha_j x_j^{k_j-1}\sum_{s_j=0}^{\infty}\left[(k_j-1)x_j^{k_j-1}\right]^{s_j} t_j^{s_j} - r_j p_j x_j^{r_j+k_j-1}\sum_{s_j=0}^{\infty}\frac{\left(\frac{r_j}{k_j-1}+1\right)_{s_j}}{s_j!}\left\{(k_j-1)x_j^{k_j-1}\right\}^{s_j} t_j^{s_j}\right]$$

$$\left[\sum_{n_1,\dots,n_m=0}^{\infty} G_{n_1,\dots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_{i-1}^{n_{i-1}}}{n_{i-1}!}\frac{t_i^{n_i-1}}{(n_i-1)!}\frac{t_{i+1}^{n_{i+1}}}{n_{i+1}!}\cdots\frac{t_m^{n_m}}{n_m!}\right]$$

$$= \left[\alpha_i x_i^{k_i-1}\sum_{s_i=0}^{\infty}\left[(k_i-1)x_i^{k_i-1}\right]^{s_i} t_i^{s_i} - r_i p_i x_i^{r_i+k_i-1}\sum_{s_i=0}^{\infty}\frac{\left(\frac{r_i}{k_i-1}+1\right)_{s_i}}{s_i!}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i} t_i^{s_i}\right]$$

$$\left[\sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_{j-1}^{n_{j-1}}}{n_{j-1}!}\frac{t_j^{n_j-1}}{(n_j-1)!}\frac{t_{j+1}^{n_{j+1}}}{n_{j+1}!}\cdots\frac{t_m^{n_m}}{n_m!}\right].$$

Hence equating the coefficients of $t_1^{n_1}\ldots t_m^{n_m}$ both the sides, we derive $m(m\text{-}1)$ recurrence relations in the following unified form :

$$(7.4)\quad \alpha_j x_j^{k_j-1}\left[(k_j-1)x_j^{k_j-1}\right]^{s_j}\frac{n_j!}{(n_j-s_j)!}G_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_{j-1},n_j-s_j,n_{j+1},\ldots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$-r_j p_j x_j^{r_j+k_j-1}\left(\frac{r_j}{k_j-1}+1\right)_{s_i}\left[(k_j-1)x_j^{k_j-1}\right]^{s_i}\binom{n_j}{s_j}$$

$$G_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_{j-1},n_j-s_j,n_{j+1},\ldots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$=\alpha_i x_i^{k_i-1}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i}\frac{n_i!}{(n_i-s_i)!}G_{n_1,\ldots,n_{i-1},n_i-s_i,n_{i+1},\ldots,n_{j-1},n_j+1,n_{j+1},\ldots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\ldots,x_m)$$

$$-r_i p_i x_i^{r_i+k_i-1}\left(\frac{r_i}{k_i-1}+1\right)_{s_i}\left\{(k_i-1)x_i^{k_i-1}\right\}^{s_i}\binom{n_i}{s_i}$$

$$G_{n_1,\ldots,n_{i-1},n_i-s_i,n_{i+1},\ldots,n_{j-1},n_j+1,n_{j+1},\ldots,n_m}^{([\alpha];[k];[r];[p])}(x_1,\ldots,x_m)\qquad i,j=1,\ldots,m;\ i\neq j.$$

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SP-QUADTREE : AN APPROACH OF DATA STRUCTURING FOR PARALLELIZATION OF SPATIAL DATA

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ABSTRACT

Emerging spatial database applications will require this availability of various index structures due to the heterogeneous collection of data types they deal with. Modern *GIS* can accept different kinds of data after resturcturing and partitioning the spatial data depending on application. This paper introduces the new domain decomposition algorithm to create a valuable set of spatial data to speed up the performance of hybrid *GIS* Data set. *SP*-quadtree is an interesting choice for spatial databases, *GIS*, and other modern database systems. In the following section concerned algorithm is examined and the experimental results have also been presented in this paper.

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1. Introduction. Complexities are involved in the access of spatial data. There are problems in handling of these spatial database in query operations in single machine and multi machine environment. A single query locks the entire spatial data to maintain data consistency for its sole processing and prohibits its access by other query process. Number of queries can run concurrently if the locks can be arranged in sub domain related to that query. There exists a possibility to partition the spatial domain data set into disjoint sub domain data sets. The data

structure in hierarchical fashion makes a provision for managing locks on sub domain. Thus a query process will lock only the relevant sub domain of spatial database leaving other sub domain to be managed by other concurrent query on several computing units.

Therefore, a detailed study is required for the common features among the members of the spatial space partitioning tecniques aiming at the capability of representing and producing the partitioned spatial domain data. Moreover, a new extensible index partitioning tecnique is required to support the class of space partitioning data structures for a domain.

2. Space-Partitioning Trees : Overview, Challenges

A single query locks the entire spatial data to maintain data consistency for its sole processing and prohibits its access by other query process. Number of queries can run concurrently if the locks can be arranged in sub domin related to that query. There exists a possibility to partition the spatial domin data set into disjoint sub domin data sets. The data structure in hierarchical fashion makes a provision for managing locks on sub domin. Thus,a query process will lock only the relevant sub domin of spatial database leaving other sub domin to be managed by other concurrent query on several computing units.

Emerging *GIS* spatial data applications require the use of new indexing structure beyond *B*+tree, *R* tree [01] [08]. The new applications need different structure to suit the big variety of spatial data e.g. multidimensional data. The space partitiong trees are well suited for such multidimensional relational spatioal data. The term space-partitioning refers to the class of hierarchical data structure that recursively decomposes a certain space into disjoint parttions. The structural and behavioral similarities among many spatial trees create the class of space-partitioning trees [03]. Space-partitioing trees can be differentiating on the structural differences (i.e. types of data, resolution, structure etc.) and behavioral differences (i.e. the decomposition principles). The main characteristic of space-partitioning trees for spatial data is that they partition the multidimensional space into disjoint (non-overlapping) regions. Partitioning can be either (1) space-driven that decomposes the space into equal-sized partitions regardless of the data distribution, its means the space is partition physically [07], (2) data-driven that splits the data set into equal portions based on some criteria, e.g., based on one of the dimensions. This refers the partition is made logically for spatial data of space [07]. So if the principle of decomposing on the input data, it is called data driven decomposition, while if it is dependent solely on the space, it is called space driven decomposition.

There are many types of trees i.e. Height Balancing Tree, *R*-trees, *SR*-trees.

KD-trees, *PMR*-trees etc. in the class of space partitioning trees that differ from each other in various ways. Some of the important variations in the context of the tree data structure are:

***Path Shrinking.** Two different types of nodes exist in any space-partitioning tree, namely Indexed node (internal nodes) and data nodes (leaf nodes) [06]. The problem is associated as how to avoid lengthy and skinny paths from a root to a leaf. Paths of one child can be collapsed into one node. The address nodes are merged together to eliminate all single child internal nodes.

***Node Shrinking.** The problem is that with space-driven partitions, some partitions may end up being empty. In this strategy no indexed node will have one leaf node as user decomposes only when there is no room for newly inserted data items.

* **Clustering.** This is one of the most serious issues when addressing disk-based space-partitioning trees. The problem is that tree nodes do not map directly to disk pages. In fact, tree nodes are usually much smaller than disk pages. Node clustering means choosing the group of nodes that will reside together in the same disk pages.

The quadtree structures, used to code spatial data, have been geometrically varied. A quadtree is a representation of a regular partitioning of space where regions are split recursively untill there is a constant amount of information contained in them. Each quadtree block (also referred to as a *cell*) covers a portion of space. Quadtree blocks may be further divided into sub-bloks of cqual size recursively. In many hierarchical partitioning of the space has been in terms of regular-shaped squares and each may be produced to different users in parallel manner. However, with some structures [06] the data themselves have been used to determine the position and shape of the subdivisions. The distribution or position of the points, lines, and vectors may be used to divide the space into irregular shape at each subdivision so helping to represent the variations in density and distribution of data values at various scales. The main problem with existing quadtrees with spatial data is the loss of explicit topological referencing [04], because of paths of sub-block can be collapsed into the block and that the precision of point and linear features are limited. So the existing quadtree structure is not well suited due to the importance of referencing for hybrid relational *GIS* Data. The new proposed *SP*-Quadtree data structuring techniques may be used with hybrid relational *GIS* data where the presence or absence of an attribute determines the coding of a quadrant for parallel *GIS* and does not loss the topological referencing.

Another problems of node shrinking in which empty node exists after some

time this becomes valuable (e.g.flood, new building) and block becomes important. This work includes the concept of data warehousing i.e. non-volatile in SP-Quadtrees and contains the previous information.

In comparisons, updating the attribute data is trivial; provide the one-to-one links between attribute data records and the spatial entities remain unaltered. Traditional database systems empoly indexes on alphanumeric data, usually based on the *B*-tree [01] [02], to facilitate effcient query handling. Typically, the database system allows the users to designate which attributes (data fields) need to be indexed. However,advanced query optimizers [14] [12] also have the ability to create indexes on un-indexed data of temporary results (i.e., results from a part of the query) as needed. In order to be worthwhile, the index creation process must not be much time-consuming as otherwise the operation could be executed more efficiently without an index. Spatial indexes such as the *SP*-quadtree are important in spatial databases for efficient execution of queries involving spatial constraints, espectially when the queries involve spatial joins. This research investigates the issue of speeding up building domain based on SP-Quadtree for a set of spatial data and develops an algorithm to achieve this goal. Spatial indexes are designed to facilitate spatial database operations that involve retrieval on the basis of the values of spatial attributes. If the database is static, user can afford to spend more time on building the index as the index creation time can be amortized over the number of queries made on the indexed data. However, this research considers the case that the database is dynamic. This situation arises when the output of an operation. In this case, evaluation of the effeciency of appropriateness of a particular spatial index must also take into account the time needed to build an index on the results of the operation. The time to bulid the spatial index plays an important role in the overall performance of hybrid relational *GIS* database.

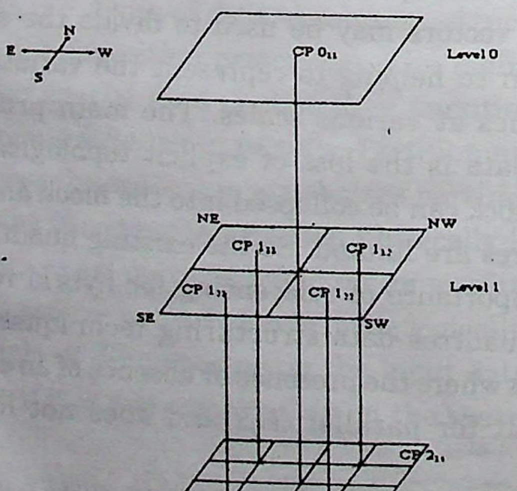

Figure 1: ***A SP*-Quadtree Implementation**

SP-Quadtrees can be implemented in many different ways. A key aspect of implemenation of the *SP*-Quadtree is its splitting rule in which regions are split recursively into quadrants. The method presented here, inspired by viewing them as trees, as shown in figure 1, in which *SP*-quadtree is two dimensional space in a 4-way branching tree that represents a recursive decomposition of space wherein at each level a square subspace is divided into four equal size squares (i.e. quad block or cell) labeled the north-east, north-west, south-east, and south-west quadrants. So the area of each quadrant at level 1 is one fouth of area of level 0.

Area $L_1 = 1/4$ Area L_0

At level 2

Area $L_2 = (1/4)^2$ Area L_0.

In general, the area or space at any level of one qualrant can be represent as

Area $L_i = (1/4)^i$ Area L_0.

Quad blocks are managed in row, column order in *SP*-quadtree. At first level ($n=0$) *SP*-quadtree contains one row ($i=1$) and one column ($j=1$), at second level ($n=1$) *SP*-quadtree contains two rows ($i=1,2$) and two columns ($j=1,2$), at third level ($n=2$) *SP*-quadtree contains four rows ($i=1,2,3,4$), and four columns ($j=1,2,3,4$), at fourth level ($n=3$) *SP*-quadtree contains eight rows ($i=1,2,...,8$) and eight columns ($j=1,2,...8$). In general at a level ($n=n$), the *SP*-quadtree contains 2^n rows (i.e. $1,2,...,2^n$) and 2^n columns (i.e. $1,2,...2^n$). Each quad block contains the control point (*CP*) at central location for maintaining the reference of next level control points and denoted as *CP* Level row column inforior (e.g. $CP\ 1_{12}$). At level zero the control point is noted as $CP0_{11}$, at level one ($n=1$) *SP*-quadtree contains a set of four control points (i.e. $CP\ 1_{11}$, $CP\ 1_{12}$, $CP1_{21}$, $CP1_{22}$). So at any level ($n=n$) there exist a set of control points, that may be represented as

$\{CPn_{ij}\}$ where $1 \le i \le 2^n$ and $1 \le j \le 2^n$.

Any control point at any level contains the referencing information of next level control points. For example at level zero, control point ($CP\ 0_{11}$) contains the information of level 1 control points as a set of four control points (i.e $CP\ 1_{11}$, $CP1_{12}$, $CP1_{21}$, $CP1_{22}$). The referencing set of control points at any level can be represented as

$CPn_{ij} = \{CPn+1_{(2i-1,2j-1)}, CPn+1_{(2i-1,2j)}, CPn+1_{(2i,2j-1)}, CPn+1_{(2i,2j)}\}$.

The location and size with control points of each child leaf block is encoded in such manner so that they do not loose the referencing and are used as a key into an auxiliary disk-based data structure. This approach is termed a *linear* SP-*Quadtree*. If control point varied from original location to another point location is termed as *Point* SP-*Quadtree*. This situation is useful when space partitioning is done on the subject oriented bases.

3. Algorithm. This section gives the algorithm with description for spatial data domain decomposition. The *SP*-quadtree node definition in our spatial domain decomposition algorithm is described as the *C* language structures. This structures starts with initiation for length and level of the domain and decides the control points. When the domain is decomposed, it forms rows and column so this structure contains the information for start and end for rows and columns.

```
typedef struct SP quadnode_struct
{
int length;/* square side length of the node*/
int level;/* a parameter to control how deep a node should be divided*/
int control_pt num;/* number of control ponts in this node*/
int startrow;/* start row number of this node in global grid*/
int startcol;/* start column number of this node in global grid*/
int endrow;/* end row number of this node in global grid*/
int endcol;/* end column number of this node in global grid*/
} SP_Quadnode;
```

For the implementation, a *SP*-Quadtree is stored in a single direction list as the following *C* language structure shows. typedef struct SP_quadnode_list_struct

```
{
SP_Quadnode quad;
struct object_list struct*next,/* Pointers to next SP_Quadnode object*/
} SP_Quadnode list node;
```

Proposed SP-quadtree-based spatial domain decomposition algorithm is designed for general use and it can produce scalable geographical workloads that can be allocated to available computational resources and speedup the performance of hybrid relational databases. The algorithm addresses the decomposition challenges [05] that are centered on finding efficient data paritition s that are assigned to each *GIS* resource. The SP_*Quad_Decompose* algorithm has two input parameters: level and *min_length*. In *SP_ Quad_Decompose*, these two parameters together with information about the number of control points at each *SP*-quadtree node are used to determine the level and location of recursive division. In the execution of *SP_Quad_Decompose*, those parts of a region that contain a high density of control points are recursively divided until the level of their SP-quadtree nodes reaches zero from a user specified value (greater than 0). However, regions with a low density of control points will not stop being divided until the square length of the node is less than the parameter *min_length*. Regions with an intermediate density of control points will invoke either of these two rules depending on which is satisfied first.

First of all the algorithm initializes the first node of *SP*-Quadtree rather than assigning the level to current node in the step 1 and 2. In the step 3, algorithm tests the conditions for splitting the node. If qual level is not greater than zero, stop the algorithm (i.e. steps 3.a.2). The algorithm is also break of quad length is not greater than minimum length through step 3.b. If current node has to be split, generate the four children for north-east, north-west, south-east and south-west regions of current quad node and assign the control points for them than current quad node has been removed. The four new children of the deleted quad node have been appended at the end of list. The algorithm is desctibed as follows in figure 2.

SP_Quad_Decompose (int *level*, int *min_length*)

1. Initialize the first node of a SP_quadtree/* curr is the pointer that is always pointing to the currently accessed node*/

2. curr-> SP_quad. level =***level***

3. while (TRUE)

{

3(a). if (curr->SP_quad.level>0)

{

3(a.1). if (curr->SP_quad.length>***min_length***)

{

3(a.1.1). Intialize four children of this node:

p_sw, p_se, p_nw, p_ne

3(a.1.2). Assign control points to four children

3(a.1.3). if (curr->SP_quad.control_pt_num>=SEARCHNUM)

/*SEARCHNUM is the *k* in *k*-nearest neighbor search*/

{

p_sw->SP_quad.level=curr->SP_quad.level-1;

p_se->SP_quad.level=curr->SP_quad.level-1;

p_nw->SP_quad.level=curr->SP_quad.level-1;

p_ne-> SP_quad.level=curr->SP_quad.level-1;

}

3(a.1.4). else

{

p_sw->SP_quad.level=curr->SP_quad.level;

p_se->SP_quad.level=curr->SP_quad.level;

p_nw->SP_quad.level=curr->SP_quad.level;

p_ne-> SP_quad.level=curr->SP_quad.level;

}

3(a.1.5). Add four children to the list in Morton order and delete the current node

```
that curr is pointing to control point.
}
3(a.2). else
if ((curr->next)=NULL)
break;
else
curr=curr->next;
}
3(b). else
if((curr->next)= NULL)
break;
else
curr=curr->next;
}
```

Figure 2: An Algorithm for *SP*-Quadtree based Domain Decomposition

The splitting module of step 3 of algorithm in pictorial manner is shown in figure 3. This points the current node which has to split than generate the four children. The current node is removed and four new children are appended at the end of list.

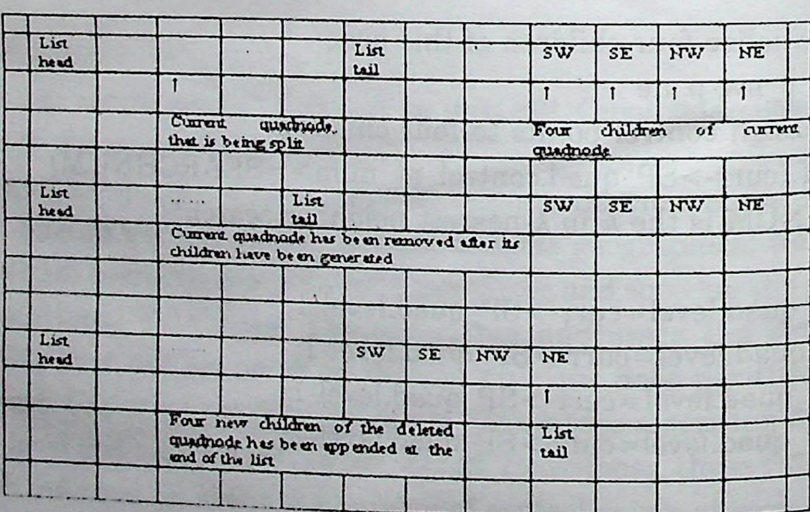

Figure 3: Formation of Quad using List in SP-Quadtree Algorithm

Using two parameters: *level* and *min_length*, the SP_*Quad_Decompose* algorithm produces balanced workloads of two types: extensive un-interpolated areas with large number of control points and compact un-interpolated areas with small (quite often, equal to zero)number of control points. However, users can specify these two parameters to determine the granularity of workloads according to the computing capacity of their accessible *GIS* resources. For example, users can increase min_length from 65 to 125 to generate larger workloads. These values,

which are specific to this 2000 by 2000 problem, represent the side length of Hybrid cells generated from the Clarke algorithm [12]. These values can easily be converted to real world units or ratios based on the size of entire *GIS* data sets.

4. Performance Evaluation. The domain decomposition algorithms descrived in the previous sections are evaluated using the four datasets on *SUN FIRE X*4200 $m2$ system of two 2.4 *Ghz* duel core *AMD* optern 2000-series processors with 4 *GB DDR*2/667 *ECC* registered memory. The system architecture contains three 8.0 *GB*/sec hyper transparent links per processor and 10.7 *GB*/sec access between each processors and memory, and has four channel *SAS* interface with *SUN RAY* server softwere with solaris 64 bit. For each dataset, two parameters in *SP_Quad Decompose* algorithm (*level* and *length*) were adjusted to determine the parameter configurations that yield the best performance (i.e., minimum computing time), and also calculated speedup S, which is defined as

$S=t1/tn$(eq no.1)

where $t1$ is the run time realized on a single processor that is selected by the broker (the selected processor is the fastest one among all avilable hardware environment); and tn is the computing time when multiple processors are used to conduct computations.

For the uniformly distributed data, the single parameter level was adjusted because *length* does not affect the process of decomposing the spatial domain. Different *level* values were chosen to observe which level value achieves the best performance. All resources are used except that when level is equal to 1, two sites are used because only four jobs are generated by the *SP_Quad_Decompose* algorithm. It was found that when level is equal. to 2,the best performance is achieved because of the characteristics of the Grid computing environment and the problem being addressed. In particular, there is an overhead penalty imposed on each job that is generated by the *SP_Quad_Decompose algorithm*. This penalty can be expressed in the following equation:

$P=Ti+Tj$(eq no.2) where Ti is the job initiation time.

Table A and figure 4 show that without decomposition it takes more time when decomposition starts on different level and different length. It reduces the computing time when at large level and length it becomes steady due to increase of decomposition overhead.

	Without Decomposition	With Decomposition					
		Level = 2		Level = 3		Level = 4	
		Length = 64	Length = 128	Length = 64	Length = 128	Length = 64	Length = 128
Computing Time (Sec)	69.48	34.47	42.08	43.16	55.17	47.34	59.58

Table A : Comparison of Computing Time for With and Without Decomposition

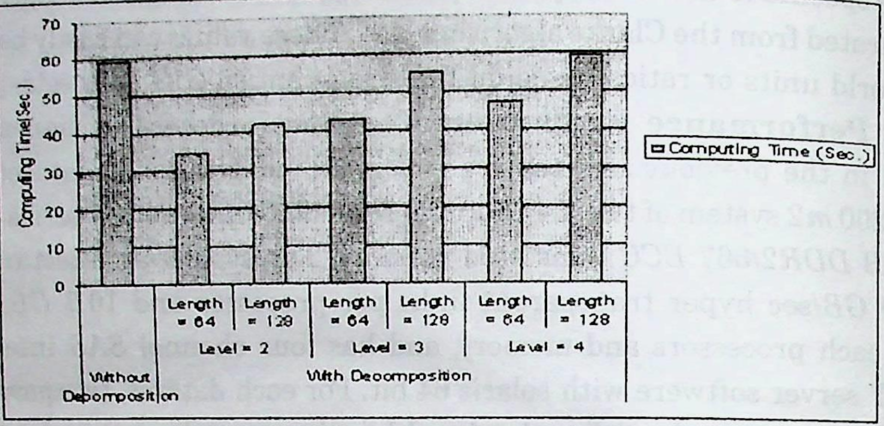

Figure 4: Comparison of computing time

Figure 5 compares the execution time when *SP*-quadtrees for the point data with that for a general quadree. The execuation time appears to grow linearly with the dimension for both algorithms. This is to be expected, since the size of the point data as well as the time to compute geometric operations grows linearly with the dimension. The quadtree algorithm is slightly faster for all dimensions but the difference between the two techniques gradually decreases as the number of dimensions increase. This difference corresponds to the overhead (in terms of execution time) in the *SP*-quadtree algorihm due to the use of the pointer-based quadtree. However, the relative parity of the two algorithms is only achieved when the improved *SP*-quadtree algorithm is used. Without it, the execution time for the *SP*-quadtree algorithm grows exponentially with the dimension.

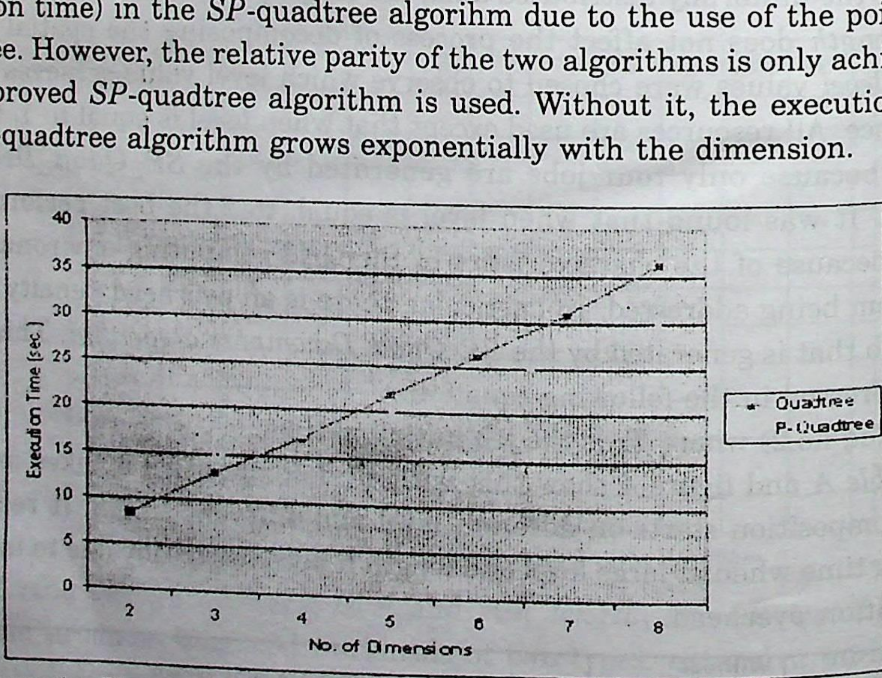

Figure 5: Execution Time for Quadtrees and SP-Quadtrees for Point Data Sets of Varying Dimensionality.

5. Conclusion. The space partitioning tree methodology is implemented in this paper. This makes it possible to have single tree index implementation to cover various types of trees that suit different applications. The general objective

of this paper is to investigate and speedup the performance of a hybrid *GIS* data set in parallel manner. A *SP*-Quadtree-based domain decomposition algorithm was developed and evaluated when used uniformly distributed and clustered datasets in the hybrid computing environment. The results show that for a dataset with a uniform random distribution,the domain decomposition algorithm scales well given the available *GIS* resources. More specifically, speedup is increased when additional resources are used. As expected, sites with larger computing capacities contributed more to observed increases in speedup. Future work may include the same methodology for more complex oprations such as spatial joins.

6. Future Scope. To improve times related to a distributed query, the final cost of the operation shoud be reduced. This cost is made of processing and communication costs. The latter one have greater impact on the final response time. Despite the reduction of the number of messages exchanged between servers, it is important to emphsize the need for an adequate operating system and network tuing when using high performance network paths [09] [10] [11] [13]. Load balancing is another big challenge to be highlighted since geographic slices produced by the broker though hot considered here but may affect the space partitioning. The work may be extended to investigate whether the buffering strategies for bulk-loading may be used to speed up dynamic insertions and queries. Also, the fact that the system can build quadtrees effciently will enable user to build a spatial query processor that exploits this to construct spatial indexes for temporary results or for un-indexed spatial relationship prior to spatial operations on them.

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LIE THEORY AND BASIC CLASSICAL POLYNOMIAL

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ABSTRACT

In the present paper, an attempt has been made to bring basic hypergeometric functions with in the perview of Lie theory by constructing a dynamical symmetry algebra of basic hypergeometric function $_2\Phi_1$. Multiplier representation theory is then used to obtain generating function for basic analogues of Gegenbauer polynomials.

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1. Introduction. The q-analogue of the Gauss function or Heine's series [1.p.259, eqn(1)] may be written as

$$_2\Phi_1(a,b;c;q;x)=\sum_{n=0}^{\infty}[a;q,n][b;q,n]x^n/[c;q,n][n;q]!$$

(where $c \neq 0,-1,-2,\ldots$, $|q|<1$ and $|x|<1$)

Here $[a;q,n]$ and $[n;q]!$ are respectively the basic Pocchammer's symbol and basic factorial function defined as $[a;q,n]=[a;q][a+1;q]..[a+n-1;q]$ and $[n;q]!=[1;q][2;q]\ \ldots[n;q]$.

The basic differential operator $B^s_{q,x}$ is defined [1, p.259, eqn (2)] by the relation

$$B^{\wedge}_{q,x}\Phi(x)=\{\Phi(qx)-\Phi(x)\}/x(q-1) \qquad \ldots(1.1)$$

2. The Dynamical Symmetry Algebra of $_2\Phi_1$. The dynamical symmetry

algebra of the hypergeometric function has been defined by Miller[2]. We use the same technique to define the dynamical symmetry algebra of ${}_2\Phi_1$.

Let $\Phi_{\alpha,\beta,\gamma,q} = \{\Gamma_q(\gamma-\alpha)\Gamma_q(\alpha)/\Gamma_q(\gamma)\} \times {}_2\Phi_1[\alpha,\beta;\gamma;q;x]s^{\alpha}u^{\beta}t^{\gamma}$ ...(2.1)

be the basis elements of a subspace of analytical functions of four variables x, s, u and t, associated with Heine's basic hypergeometric fucntions of Heine's series, ${}_2\Phi_1$. Introduction of variables s, u and t renders differential operators independent of parameters α,β and γ and thus facilitates their repeated operation.

The dynamical symmetry algebra of ${}_2\Phi_1$ is a 15-dimensional complex Lie algebra isomorphic to $sl(4)$, generated by twelve $E^{\wedge}$-operators termed as raising or lowering the corresponding suffix in $\Phi_{\alpha,\beta,\gamma,q}$.

The $E^{\wedge}$-operators is $E^{\wedge}_{-\alpha,q} = s^{-1}\left(x(1-x)B^{\wedge}_{q,x} + TB^{\wedge}_{q,t} - sB^{\wedge}_{q,s} - xuB^{\wedge}_{q,u}\right)$ ...(2.2)

The action of this operator on $\Phi_{\alpha,\beta,\gamma,q}$ is given by

$E^{\wedge}_{-\alpha,q}\Phi_{\alpha,\beta,\gamma,q} = [\alpha-1;q]\Phi^{1}_{\alpha,q,\beta,\gamma,q}$ ...(2.3)

Twelve $E^{\wedge}$-operators together with three maintenance operators $J_{\alpha}, J_{\beta}, J_{\gamma}$ and Identity operator I form a basis for $gl(4) \cong sl(4)(I)$, where (I) is the 1-dimensional Lie algebra generated by I.

Here $J^{\wedge}_{\alpha,q} = sB^{\wedge}_{q,x}; J^{\wedge}_{\beta,q} = uB^{\wedge}_{q,u}; J^{\wedge}_{\gamma,q} = tB^{\wedge}_{q,t}$ and $I^{\wedge} = I$ ...(2.4)

with the results

$J^{\wedge}_{\alpha,q}\Phi_{\alpha,\beta,\gamma,q} = [\alpha;q]\Phi_{\alpha,\beta,\gamma,q}; J^{\wedge}_{\beta,q}\Phi_{\alpha,\beta,\gamma,q} = [\beta;q]\Phi_{\alpha,\beta,\gamma,q}$

$J^{\wedge}_{\gamma,q}\Phi_{\alpha,\beta,\gamma,q} = [\gamma;q]\Phi_{\alpha,\beta,\gamma,q}$ and $I^{\wedge}\Phi_{\alpha,\beta,\gamma,q} = \Phi_{\alpha,\beta,\gamma,q}$ (2.5)

3. The Generating Function for Basic Analogues of Gegenbauer Polynomial. The action of one parameter subgroup ($\exp_q aE^{\wedge}_{-\alpha,q}$) generated by the operator $E^{\wedge}_{-\alpha,q}$ defined in (2.2) on $\Phi_{\alpha,\beta,\gamma,q}$ defined in (2.1) can be computed by the multiplier representation method.

Using the technique of Miller [3] it can be seen that the transformations

$x \to xs/(a(x-1)+s), s \to s-a, u \to u(s-a)/(a(x-1)), t \to st/(s-a)$ ...(3.1)

determine the action.

Hence the action of one parameter subgroup is given by

$$\left(\exp_q aE^{\wedge}{}_{-\alpha,q}\right)\Phi_{\alpha,\beta,\gamma,q} = \{\Gamma_q(\gamma-\alpha)\Gamma_q(\alpha)/\Gamma_q(\gamma)\}\times {}_2\Phi_1[\alpha,\beta;\gamma;q;xs/(a(x-1)+s)]$$

$$\times(s-a)^{\alpha}(u(s-a)/(a(x-1)))^{\beta}(st/(s-a))^{\gamma} \quad ...(3.2)$$

On the other hand by expansion, we get

$$\left(\exp_q aE^{\wedge}{}_{-\alpha,q}\right)\Phi_{\alpha,\beta,\gamma,q} = \sum_{m=0}^{\infty} a^m[\alpha-m;q]_m/[m;q]!\times\Gamma_q(\gamma-\alpha+m)\Gamma_q(\alpha-m)/\Gamma_q(\gamma)$$

$$\times {}_2\Phi_1\left[\alpha q^{-m},\beta;\gamma;q;x\right]s^{\alpha-m}u^{\beta}t^{\gamma} \quad (3.3)$$

Equating these two values of $\left(\exp_q aE^{\wedge}{}_{-\alpha,q}\right)\Phi_{\alpha,\beta,\gamma,q}$ we get identity

$$(s-a)^{\alpha+\beta-\gamma}(a(x-1)+s)^{-\beta}s^{\gamma-\alpha}\times {}_2\Phi_1[\alpha,\beta;\gamma;q;xs/(a(x-1)+s)]$$

$$=\sum_{m=0}^{\infty}\left\{a^m[\gamma-\alpha;q]_m/[m;q]!\right\}\times {}_2\Phi_1\left[\alpha q^{-m},\beta;\gamma;q;x\right]s^{-m} \quad ...(3.4)$$

For example, Putting $\alpha\to 0, s\to 1, \beta\to\lambda+m, \gamma\to 1/2, q\to q^2$ and then $x\to x^2$, we get

$$(1-a)^{\lambda-1/2}\left(1-a+ax^2\right)^{-\lambda} = \sum_{m=0}^{\infty}\left\{a^m[1/2;q]_m[1/2;q]/[m;q]!\right\}\times {}_2\Phi_1\left[-m,\lambda+m;1/2;q^2;x^2\right]. \quad (3.5)$$

By the definition of basic Gegenbauer polynomials [4]

$$C_{2m}^{\lambda}(q;x) = {}_2\Phi_1\left[-m,\lambda+m;1/2;q^2;x^2\right], \quad (3.6)$$

Using (3.6) in (3.5), we get the generating function

$$(1-a)^{\lambda-1/2}\left(1-a+ax^2\right)^{-\lambda} = \sum_{m=0}^{\infty}\left\{a^m[1/2;q]_m[1/2;q]/[m;q]!\right\}\times C_{2m}^{\lambda}(q;x) \quad (3.7)$$

for basic Gegenbauer polynomials.

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